- HW length
- Please tag your forum posts.
- Ask me oo @ 12:30 after class
- Examlet 1 : book your time
- HW3
- HW sol is shown after due ends

$$\|A\|_2 = \max_{\|x\|=1} \|Ax\|_2$$

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

- $\|A\| > 0 \Leftrightarrow A \neq 0$. definiteness
- $\|\gamma A\| = |\gamma| \, \|A\|$ for all scalars γ. semilin
- Obeys triangle inequality $\|A + B\| \leq \|A\| + \|B\|$ Δ-ineq.

But also some more properties that stem from our definition:

$$\|Ax\| \leq \|A\| \, \|x\|$$

$$\|AB\| \leq \|A\| \, \|B\|$$

submultiplicativity

$A \Delta x = \Delta b \quad | A^{-1}\cdot$

$\Delta x = A^{-1} \Delta b$

Conditioning

What is the condition number of solving a linear system $A\boldsymbol{x} = \boldsymbol{b}$?

Input: $\vec{b}$ with error $\Delta\vec{b}$

Output: $\vec{x}$ with error $\Delta\vec{x}$

$$A(\vec{x} + \Delta\vec{x}) = \vec{b} + \Delta\vec{b} \Rightarrow A\Delta\vec{x} = \Delta\vec{b}$$

$$\frac{\text{rel. error in output}}{\text{rel. error in input}} = \frac{\|\Delta x\| / \|x\|}{\|\Delta b\| / \|b\|} = \frac{\|\Delta x\| \; \|b\|}{\|\Delta b\| \; \|x\|}$$

$$= \frac{\|A^{-1}\Delta b\| \; \|Ax\|}{\|\Delta b\| \; \|x\|}$$

$$\leq \|A\| \|A^{-1}\|$$

Conditioning of Linear Systems: Observations

Showed $\kappa(\text{Solve } A\boldsymbol{x} = \boldsymbol{b}) \leq \left\|A^{-1}\right\| \|A\|$.
I.e. found an *upper bound* on the condition number. With a little bit of fiddling, it's not too hard to find examples that achieve this bound, i.e. that it is *sharp*. ✓ (see HW)

So we've found the *condition number of linear system solving*, also called the condition number of the matrix A:

$$\operatorname{cond}(A) = \kappa(A) = \underline{\underline{\|A\| \left\|A^{-1}\right\|}}.$$

rel. to the (matrix) norm.

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2$$

Conditioning of Linear Systems: More properties

- cond is relative to a given norm. So, to be precise, use

$$\text{cond}_2 \quad \text{or} \quad \text{cond}_\infty .$$

- If A^{-1} does not exist: $\text{cond}(A) = \infty$ by convention.

recall: square

What is $\kappa(A^{-1})$?

$\kappa(A)$

What is the condition number of matrix-vector multiplication?

solve: $Ax = b$ matvec: $A\vec{x} = \vec{b} \iff A^{-1}\vec{b} = \vec{x}$

equiv. solving lin. system

Demo: Condition number visualized [cleared]
Demo: Conditioning of 2x2 Matrices [cleared]

Residual Vector

What is the residual vector of solving the linear system

$$\boldsymbol{b} = A\boldsymbol{x}?$$

$$\vec{r} = \vec{b} - A\hat{\vec{x}}$$

computable!

Residual and Error: Relationship

How do the (norms of the) residual vector $\boldsymbol{r}$ and the error $\Delta\boldsymbol{x} = \boldsymbol{x} - \widehat{\boldsymbol{x}}$ relate to one another?

$$\|\Delta\vec{x}\| = \|\vec{x} - \hat{\vec{x}}\| = \|A^{-1}(b - A\hat{\vec{x}})\| = \|A^{-1}\vec{r}\| \quad \text{computable!}$$

$$\underbrace{\frac{\|\Delta\vec{x}\|}{\|\hat{\vec{x}}\|}}_{\text{"rel. error"}} = \frac{\|A^{-1}\vec{r}\|}{\|\hat{\vec{x}}\|} \le \frac{\|A^{-1}\|\|\vec{r}\|}{\|\hat{\vec{x}}\|} \overset{\cdot\frac{\|A\|}{\|A\|}}{=} \operatorname{cond}(A)\frac{\|\vec{r}\|}{\|A\|\|\hat{\vec{x}}\|} \le \operatorname{cond}(A)\underbrace{\frac{\|\vec{r}\|}{\|A\hat{\vec{x}}\|}}_{\text{rel. resid}}$$

$$\|A\|\,\|\hat{\vec{x}}\| \ge \|A\hat{\vec{x}}\|$$

$$\text{"rel. error"} \le \operatorname{cond}(A) \cdot \text{rel. resid}$$

Changing the Matrix

So far, only discussed changing the RHS, i.e. $A\boldsymbol{x} = \boldsymbol{b} \quad \rightarrow \quad A\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{b}}$.
The matrix consists of FP numbers, too—it, too, is approximate. I.e.

$$A\boldsymbol{x} = \boldsymbol{b} \quad \rightarrow \quad \widehat{A}\widehat{\boldsymbol{x}} = \boldsymbol{b}.$$

What can we say about the error due to an approximate matrix?

$$\Delta x = \hat{x} - x = A^{-1}(A\hat{x} - b) = A^{-1}(A\hat{x} - \hat{A}\hat{x}) = A^{-1}\Delta A \hat{x}$$

$$\|\Delta x\| \le \|A^{-1}\| \|\Delta A\| \|\hat{x}\| \cdot \frac{\|A\|}{\|A\|}$$

$$\frac{\|\Delta x\|}{\|\hat{x}\|} \le \operatorname{cond}(A) \cdot \frac{\|\Delta A\|}{\|A\|}$$

"rel. error"

Changing Condition Numbers

Once we have a matrix A in a linear system $A\boldsymbol{x} = \boldsymbol{b}$, are we stuck with its condition number? Or could we improve it?

What is this called as a general concept?

In-Class Activity: Matrix Norms and Conditioning

In-class activity: Matrix Norms and Conditioning

Singular Value Decomposition (SVD)

What is the *Singular Value Decomposition* of an $m \times n$ matrix?

$A \in \mathbb{R}^{m \times n}$

$$A = U \Sigma V^T$$

col U = left. sig. vct. U : orthogonal $\leftarrow m \times m$

singular values Σ : diagonal, has nonneg. entries $m \times n$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$

$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$

col V = right. sig. vct. V : orthogonal $\leftarrow n \times n$

Computing the 2-Norm

Using the SVD of A, identify the 2-norm.

Let Q be orthogonal: $\|Q\vec{x}\|_2 = \|\vec{x}\|_2$

$\Rightarrow \|QB\|_2 = \|B\|_2 \Leftarrow$

$$\|A\|_2 = \|\underbrace{U}_{\text{orth.}} \Sigma \underbrace{V^T}_{\text{orth.}}\|_2 = \|\Sigma\|_2 = \sigma_1$$

Express the matrix condition number $\text{cond}_2(A)$ in terms of the SVD:

$\text{cond}_2(A) = \sigma_1 / \sigma_n$

Not a matrix norm: Frobenius

The 2-norm is very costly to compute. Can we make something simpler?

What about its properties?