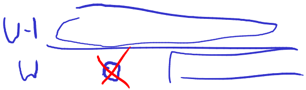
- Example 1 :



Changing Condition Numbers

Once we have a matrix A in a linear system $A\boldsymbol{x} = \boldsymbol{b}$, are we stuck with its condition number? Or could we improve it?

$D A \vec{x} = D \vec{b}$ non-sing. diag D

$M A \vec{x} = M \vec{b}$ → "left preconditioning"

right precond.: $\vec{x} = M \vec{y}$

$(A M) \vec{y} = \vec{b}$

What is this called as a general concept?

preconditioning

common example: rescale rows/cols

Not a matrix norm: Frobenius

The 2-norm is very costly to compute. Can we make something simpler?

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$

What about its properties?

- definiteness
- semilinearity
- triangle ineq
- submultiplicative

Frobenius Norm: Properties

bit.ly/cs450-F22

Is the Frobenius norm induced by any vector norm?

$\|I\|_F = \sqrt{n}$ For any induced norm, $\|I\| = 1$

How does it relate to the SVD?

$$\|A\|_F = \|\vec{\sigma}\|_2$$

$$\|A\| \|A^{-1}\| \geq \|\underline{AA^{-1}}\| = \|I\| = 1$$

$$\begin{matrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{matrix}$$

✓

$$\begin{pmatrix} \boxed{\vec{a}_1} & \boxed{\vec{a}_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2$$

$$|x_1| + |x_2| = 1$$

Solving Systems: Simple cases

Solve $D\boldsymbol{x} = \boldsymbol{b}$ if D is diagonal. (Computational cost?)

$x_i = b_i / d_{ii}$ $O(n)$

Solve $Q\boldsymbol{x} = \boldsymbol{b}$ if Q is orthogonal. (Computational cost?)

$\vec{x} = Q^T \vec{b}$ $O(n^2)$

Given SVD $A = U\Sigma V^T$, solve $A\boldsymbol{x} = \boldsymbol{b}$. (Computational cost?)

Compute $z = U^T b$

Solve $\Sigma y = z$

Compute $x = V y$ $\rightarrow$ $U \Sigma V^T x = \vec{b}$

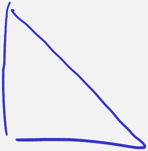
Solving Systems: Triangular matrices

Solve

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

back subst.

Fwd subst



$a_{44} w = b_4$

$a_{33} z + a_{34} w = b_3 \Leftrightarrow z = (b_3 - a_{34} w)/a_{33}$

Cost: $O(n^2)$

Demo: Coding back-substitution [cleared]

What about non-triangular matrices?

Gaussian elim.

In-Class Activity: Matrix Norms and Conditioning

In-class activity: Matrix Norms and Conditioning

Gaussian Elimination

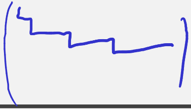
Demo: Vanilla Gaussian Elimination [cleared]

$(A \mid b)$

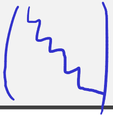
What do we get by doing Gaussian Elimination?

Row Echelon Form

How is that different from being upper triangular?



row echelon → rank-revealing



What if we do not just eliminate downward but also upward?

Gauss-Jordan elim: more expensive than LU

LU Factorization

What is the LU factorization?

$A = LU$

Lower L

Upper U

Solving $A\boldsymbol{x} = \boldsymbol{b}$

Does LU help solve $A\boldsymbol{x} = \boldsymbol{b}$?

$$A\vec{x} = \vec{b}$$

$$L\underbrace{U\vec{x}}_{\vec{y}} = b$$

$$L\vec{y} = \vec{b}$$

$$U\vec{x} = \vec{y}$$

Determining an LU factorization

LU of size $n \times n$ $\longrightarrow$

$$\begin{pmatrix} a_{11} & a_{12}^T \\ \vec{a}_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ & U_{22} \end{pmatrix}$$

scalar

vec

$$\begin{pmatrix} u_{11} & u_{12}^T \\ 0 & U_{22} \end{pmatrix} U$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ \vec{l}_{21} & L_{22} \end{pmatrix}}_{L} \begin{pmatrix} a_{11} & a_{12}^T \\ \vec{a}_{21} & A_{22} \end{pmatrix}$$

$a_{11} = u_{11}$

$\vec{u}_{12}^T = a_{12}^T$

$\vec{a}_{21} = u_{11} \cdot \vec{l}_{21} \Leftrightarrow \vec{l}_{21} = \vec{a}_{21}/u_{11}$

$A_{22} = l_{21} u_{12}^T + L_{22} U_{22}$

$L_{22} U_{22} = \underbrace{A_{22} - l_{21} u_{12}^T}_{\checkmark}$

$\hookrightarrow$ LU of size $(n-1) \times (n-1)$

Demo: LU Factorization [cleared]

Computational Cost

What is the computational cost of multiplying two $n \times n$ matrices?

- $u_{11} = a_{11}$, $\boldsymbol{u}_{12}^T = \boldsymbol{a}_{12}^T$.
- $\boldsymbol{l}_{21} = \boldsymbol{a}_{21}/u_{11}$.
- $L_{22}U_{22} = A_{22} - \boldsymbol{l}_{21}\boldsymbol{u}_{12}^T$.

What is the computational cost of carrying out LU factorization on an $n \times n$ matrix?

Demo: Complexity of Mat-Mat multiplication and LU [cleared]