Power it: (normalized)

$\tilde{x}_{k+1} = A x_k$

$x_{k+1} = \tilde{x}_{k+1} / \|\tilde{x}_{k+1}\|_2$

Shift-invert:

$A \mapsto (A - \sigma I)^{-1}$

RQI: $\sigma_k = \dfrac{x_k^T A x_k}{x_k^T x_k}$

"linear" convergence: $e_{k+1} \simeq \alpha \cdot e_k$ $\quad \alpha < 1$

"quadratic" conv. $e_{k+1} \approx \alpha e_{k+1}^2$

Rayleigh Quotient Iteration

Describe *Rayleigh Quotient Iteration*.

(see above)

Demo: Power Iteration and its Variants [cleared]

Schur form

$A = XDX^{-1}$

Show: Every matrix is orthonormally similar to an upper triangular matrix, i.e. $A = QUQ^T$. This is called the Schur form or Schur factorization.

Assume A non-defective. Let $\vec{v}$ be an eigenvector.

Build a basis: $(\vec{v}_1, \vec{w}_2 \cdots, \vec{w}_n)$ so that $\vec{v} \perp \vec{w}_i$.

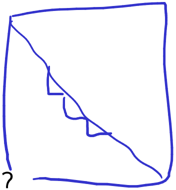
$$A = \overbrace{\begin{bmatrix} | & & \\ \vec{v} & \underbrace{\vec{w}_2 \cdots \vec{w}_n}_{\text{basis of } \vec{v}^{\perp}} \\ | & & \end{bmatrix}}^{Q_1} \begin{bmatrix} \lambda & * \\ 0 & \\ \vdots & \boxed{} \\ 0 & \end{bmatrix} Q_1^T$$

$\boxed{}$ → shares A's eigenvalues minus one multiplicity of λ

$\iff Q_1^T A Q_1 = U_1$

$$U = Q_n^T \cdots Q_2^T Q_1^T A Q_1 Q_2 \cdots Q_n$$

Schur Form: Comments, Eigenvalues, Eigenvectors



$A = QUQ^T$. For complex λ:

- Either complex matrices, or
- 2×2 blocks on diag. → complex

If we had a Schur form of A, how can we find the eigenvalues?

diag of U.

And the eigenvectors?

$$U - \lambda I = \begin{pmatrix} U_{11} & \vec{u} & U_{13} \\ 0 & \lambda - \lambda & \vec{v}^T \\ & & U_{31} \end{pmatrix}$$

$A \quad Q \quad U \quad Q^T$

$Ux = \lambda x$

$\vec{y} = Qx$

$A\vec{y} = Q\underbrace{UQ^TQx}_{\lambda x} = \lambda \vec{y}$

$x = (U_{11}^{-1}\vec{u}, -1, 0)^T$

$(U - \lambda I)\vec{x} = \vec{0}$

Qx can be evaluated at $O(n^2)$ cost.

Computing Multiple Eigenvalues

All Power Iteration Methods compute one eigenvalue at a time.
What if I want *all* eigenvalues?

- "Deflation": Find similarity transform to

$$\begin{pmatrix} \lambda & * \\ 0 & \\ \vdots & B \\ 0 & \end{pmatrix}$$

Rinse, repeat on B.

- Simultaneous iteration...?

Simultaneous Iteration

What happens if we carry out power iteration on multiple vectors simultaneously?

Pick $X_0 \in \mathbb{R}^{n \times p}$ random

$X_{k+1} = A X_k$

(probably needs some normalization)

... silly ... X_k get ill-conditioned

because all columns approx. x_1, where $Ax_1 = \lambda_{max} x_1$.

Orthogonal Iteration

rand. $X_0 \in \mathbb{R}^{n \times p}$ $(p \leq n)$

$$\hat{Q}_k R_k = X_k$$

$$X_{k+1} = A \hat{Q}_k$$

Toward the QR Algorithm

$\hat{Q}_0 R_0 = X_0$

$X_1 = A\hat{Q}_0$

$\hat{Q}_1 R_1 = X_1 = A\hat{Q}_0 \Rightarrow \hat{Q}_1^T A \hat{Q}_0 = R_1$

$\Rightarrow \hat{Q}_2^T A \hat{Q}_1 = R_2$

If $\hat{Q}_k$ converge, $\hat{Q}_{k+1} \approx \hat{Q}_k$. $\rightarrow \hat{Q}_k R_k \hat{Q}_k^T \approx A$

$\hat{X}_k := \hat{Q}_k^T A \hat{Q}_k \approx R_k$

closeness of $\hat{X}_k$ to ∇ could be used to check conv.

Demo: Orthogonal Iteration [cleared]

QR Iteration/QR Algorithm

orth. it

$X_0 = A$

$\hat{Q}_k R_k = X_k$

$X_{k+1} = A \hat{Q}_k$

QR iteration/QR algorithm

$\bar{X}_0 = A$

$\bar{Q}_k \bar{R}_k = \bar{X}_k$

$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k$

$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k = \bar{Q}_k^T \bar{X}_k \bar{Q}_k \Rightarrow \bar{X}_k$ all similar to A

$\Rightarrow$ have same eigenvalues

$A^2 = \bar{Q}_0 \bar{R}_0 \bar{Q}_0 \bar{R}_0 = \bar{Q}_0 \bar{Q}_1 \bar{R}_1 \bar{R}_0$

claim: $\hat{Q}_k$ in orth. it can be chosen to be $\hat{Q}_k = \bar{Q}_0 \bar{Q}_1 \bar{Q}_2 \cdots \bar{Q}_k$

$\hat{X}_k = \bar{X}_{k+1}$

Proof sketch: Equivalence of QR iteration/Orth. iteration

Orthogonal Iteration (no bars)

- $X_0 := A$
  - $Q_0 R_0 := X_0$,
  - where we may choose $Q_0 = \bar{Q}_0$
  - $\hat{X}_0 = Q_0^H A Q_0 = Q_0^H Q_0 R_0 Q_0 = R_0 Q_0$
- $X_1 := A Q_0$
  - $Q_1 R_1 := X_1$,
    and because of $X_1 = Q_0 Q_0^H A Q_0 = Q_0 \bar{X}_1 = Q_0 \bar{Q}_1 \bar{R}_1$
    we may choose $Q_1 = Q_0 \bar{Q}_1 = \bar{Q}_0 \bar{Q}_1$.
- $\vdots$

QR Iteration (with bars)

- $\bar{X}_0 := A$
  - $\bar{Q}_0 \bar{R}_0 := A$
- $\bar{X}_1 := \bar{R}_0 \bar{Q}_0 = \hat{X}_0$
  - $\bar{Q}_1 \bar{R}_1 := \bar{X}_1$
- $\bar{X}_2 := \bar{R}_1 \bar{Q}_1$
  - $\bar{X}_2 = Q_1^H A Q_1 = \hat{X}_1$
- $\vdots$

QR Iteration: Forward *and* Inverse

QR iteration may be viewed as performing inverse iteration. How?