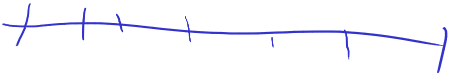
- Examlet 5: optimization up to & including equality-constrained

Representing functions:

- point values on a grid ("nodal")
- symbolically
- polynomial $a + bx + cx^2 + dx^3 + \dots$
  ↳ coefficients w/ respect to a basis
- roots + scaling (polynomials)

("modal")

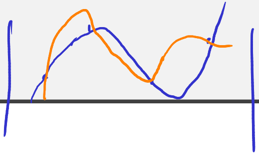


Interpolation: Setup

Given: $(x_i)_{i=1}^N$, $(y_i)_{i=1}^N$
Wanted: Function f so that $f(x_i) = y_i$

How is this not the same as function fitting? (from least squares)

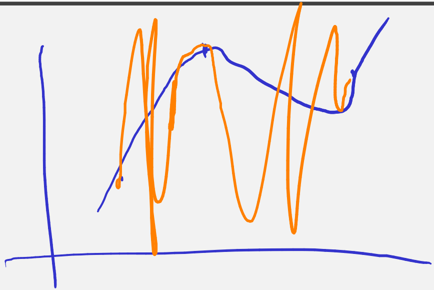
- we're asking For asking for equality
  → no residual
  → better error result
  ↳ $\| f - z \|_\infty$
  ↳ interpolant
  ↳ p_{n-1}



Interpolation: Setup (II)

Given: $(x_i)_{i=1}^N$, $(y_i)_{i=1}^N$
Wanted: Function f so that $f(x_i) = y_i$

Does this problem have a unique answer?



↳ resolve by introducing a basis

Interpolation: Importance

Why is interpolation important?

↳ calculus

↳ interpolate

↳ do the thing (e.g. derivative)

↳ evaluate

Making the Interpolation Problem Unique

interpolant $\quad p_{n-1}(x) = \sum_{j=1}^{n} \alpha_j \varphi_j(x)$ $\qquad$ One choice: $\varphi_j(x) = x^j$

what basis?

Find coefficients (α_j) via a linear system from nodes $(x_i)_{i=1}^{n}$

$$y_i = f(x_i) = p_{n-1}(x_i) = \sum_{j=1}^{n} \alpha_j \varphi_j(x_i)$$

$$V\vec{\alpha} = \vec{y} \qquad V = \big(\varphi_j(x_i)\big)_{i,j}$$

with monomials: Vandermonde matrix

with gen. basis: generalized Vandermonde matrix

$$\vec{x} = (x_i)_{i=1}^{n}$$

$$V = \left(\varphi_j(x_i)\right)_{i,j=1}^{n}$$

$$V = \begin{pmatrix} \varphi_1(x_1) & \varphi_2(x_1) & \\ \varphi_1(x_2) & & \\ & & \end{pmatrix}$$

nodes

basis

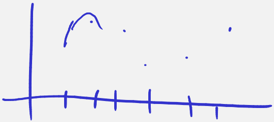
Existence/Sensitivity

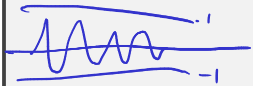
Solution to the interpolation problem: Existence? Uniqueness?

Existence/uniqueness for the linear system

Sensitivity?

- shallow answer: condition number of the linear system





Λ: Lebesgue constant

$$\max_{x\in[a,b]} |p_{n-1}(x)| \le \Lambda \|y\|_\infty$$

Modes and Nodes (aka Functions and Points)

Both function basis and point set are under our control. What do we pick?

Ideas for basis functions:

- Monomials $1, x, x^2, x^3, x^4, \ldots$
- Functions that make $V = I \to$ 'Lagrange basis'
- Functions that make V triangular $\to$ 'Newton basis'
- *Splines* (piecewise polynomials)
- *Orthogonal polynomials*
- Sines and cosines
- 'Bumps' (*'Radial Basis Functions'*)

Ideas for points:

- Equispaced
- *'Edge-Clustered'* (so-called Chebyshev/Gauss/... nodes)

Specific issues:

- Why *not* monomials on equispaced points?
  Demo: Monomial interpolation [cleared]
- Why not equispaced?
  Demo: Choice of Nodes for Polynomial Interpolation [cleared]

Lagrange Interpolation

Find a basis so that $V = I$, i.e.

$$\varphi_j(x_i) = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Lagrange Polynomials: General Form

$$\varphi_j(x) = \frac{\prod_{k=1, k\neq j}^{m}(x - x_k)}{\prod_{k=1, k\neq j}^{m}(x_j - x_k)}$$

Write down the Lagrange interpolant for nodes $(x_i)_{i=1}^{m}$ and values $(y_i)_{i=1}^{m}$.

Newton Interpolation

Find a basis so that V is triangular.

Why not Lagrange/Newton?

Better conditioning: Orthogonal polynomials

What caused monomials to have a terribly conditioned Vandermonde?

What's a way to make sure two vectors are *not* like that?

→ orthogonal

But polynomials are functions!

Orthogonality of Functions

How can functions be orthogonal?

$$\vec{f} \cdot \vec{g} = \sum_{i=1}^{n} f_i g_i = \langle \vec{f}, \vec{g} \rangle$$

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$$

Constructing Orthogonal Polynomials

How can we find an orthogonal basis?

Demo: Orthogonal Polynomials [cleared] — Got: Legendre polynomials.

But how can I practically compute the Legendre polynomials?

$$\langle f, g \rangle_w = \int w(x) f(x) g(x) \, dx$$

Chebyshev Polynomials: Definitions

Three equivalent definitions:

- Result of Gram-Schmidt with weight $1/\sqrt{1-x^2}$. What is that weight?

  (Like for Legendre, you won't exactly get the standard normalization if you do this.)
- $T_k(x) = \cos(k \cos^{-1}(x))$
- $T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$ plus $T_0 = 1,\ T_0 = x$