- hw 13
- 4CH assignment
- examlet 5 page grades

$f \in C^n$ $\quad \{x_i\}$

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!}(x-x_1)(x-x_2)\cdots(x-x_n)$$

Error Result: Simplified Form

Boil the error result down to a simpler form.

interval

$$\left|f^{(n)}(x)\right| \le M \qquad (x \in I)$$

interval length $|I| = h = b - a$

$(x - x_i)$: If $x \in I$, $|x - x_i| \le h$

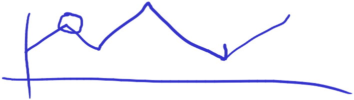
$$\text{Error}(h) = |f(x) - p_{n-1}(x)| \le CM \cdot h^n$$

$$\text{Error} = O(h^n) \qquad (\text{as } h \to 0)$$

[n-th order convergence

- **Demo:** Interpolation Error [cleared]
- **Demo:** Jump with Chebyshev Nodes [cleared]

Going piecewise: Simplest Case

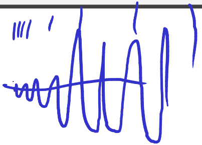


Construct a pieceweise linear interpolant at four points.

x_0, y_0		x_1, y_1		x_2, y_2		x_3, y_3
	$f_1 = a_1x + b_1$		$f_2 = a_2x + b_2$		$f_3 = a_3x + b_3$	
	2 unk.		2 unk.		2 unk.	
	$f_1(x_0) = y_0$		$f_2(x_1) = y_1$		$f_3(x_2) = y_2$	
	$f_1(x_1) = y_1$		$f_2(x_2) = y_2$		$f_3(x_3) = y_3$	
	2 eqn.		2 eqn.		2 eqn.	
	end		middle		end	

Why three intervals?

2 ends + arbitrary number of middle



Piecewise Cubic ('Splines')

n middle

x_0, y_0 | x_1, y_1 | x_2, y_2 | x_3, y_3

f_1: $f_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1$

f_2: $a_2x^3 + b_2x^2 + c_2x + d_2$

f_3: $a_3x^3 + b_3x^2 + c_3x + d_3$

12 unk

4 unkn | 4 unknown | 4 unknowns

6

$f_1(x_0) = y_0$
$f_1(x_1) = y_1$

$f_2(x_1) = y_1$
$f_2(x_2) = y_2$

$f_3(x_2) = y_1$
$f_3(x_3) = y_3$

4

$f_1'(x_1) = f_2'(x_1)$
$f_1''(x_1) = f_2''(x_1)$

$f_2'(x_2) = f_3'(x_2)$
$f_2''(x_2) = f_3''(x_2)$

"natural"

2 $f''(x_0) = 0$ $f''(x_3) = 0$

alt': $f''(x_0)$ $f''(x_3)$

Piecewise Cubic ('Splines'): Accounting ·

x_0, y_0 | x_1, y_1 | x_2, y_2 | x_3, y_3

f_1	f_2	f_3
$a_1 x^3 + b_1 x^2 + c_1 x + d_1$	$a_2 x^3 + b_2 x^2 + c_2 x + d_2$	$a_3 x^3 + b_3 x^2 + c_3 x + d_3$

Outline

$$f(x) \approx p_{n-1}(x) = \sum \alpha_i \varphi_i(x)$$

$$f'(x) \approx p_{n-1}'(x) = \sum \alpha_i \varphi_i'(x)$$

$$\int_a^b f(x)\,dx \approx \int_a^b p_{n-1}(x)\,dx = \sum \alpha_i \int_a^b \varphi_i(x)\,dx$$

Numerical Integration: About the Problem

What is numerical integration? (Or quadrature?)

Given a, b, f, approximate $\int_a^b f(x)\,dx$

What about existence and uniqueness?

- f integrable (Riemann, Lebesgue)
- f is unique for pw. continuous/bounded

Conditioning

Derive the (absolute) condition number for numerical integration.

$$\tilde{f}(x) = f(x) + e(x)$$

$$\left| \int_a^b f(x)\,dx - \int_a^b \tilde{f}(x)\,dx \right|$$

$$= \left| \int_a^b e(x)\,dx \right| \le \int_a^b |e(x)| \le (b-a) \max_{x \in [a,b]} |e(x)|$$

Interpolatory Quadrature

Design a quadrature method based on interpolation.

$$f(x) \approx \sum_{i=1}^{n} \alpha_i \varphi_i(x)$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \alpha_i \int_a^b \varphi_i(x)$$

expensive because $\vec{\alpha}$ needs to be found

Goal: $\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$

Interpolatory Quadrature: Examples

Let ℓ_i be the Lagrange polynomials for nodes $\{x_i\}_{i=1}^{n}$.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \underset{\text{nodes}}{f(x_i)} \underbrace{\int_a^b \ell_i(x)}_{\substack{w_i \\ \text{weights.}}}$$

equispaced: Newton-Cotes quadrature

Chebyshev nodes: Clenshaw-Curtis quadrature
(+ poly basis)

Interpolatory Quadrature: Computing Weights

How do the weights in interpolatory quadrature get computed?

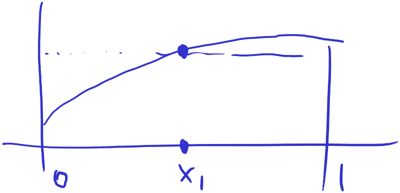
"Method of undetermined coefficients":

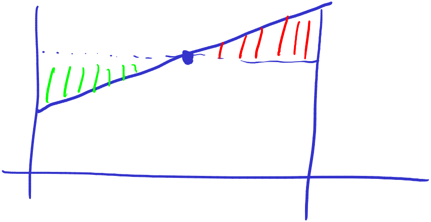
$$\vec{I} \quad \begin{pmatrix} b-a \\ \vdots \\ \frac{1}{n}(b^n-a^n) \end{pmatrix} \begin{aligned} &= \int_a^b 1\,dx = w_1 \cdot 1 + \cdots + w_n \cdot 1 \\ &= \int_a^b x\,dx = w_1 \cdot x_1 + \cdots + w_n \cdot x_n \\ &\vdots \\ &= \int_a^b x^{n-1}\,dx = w_1 x_1^{n-1} + \cdots + w_n \cdot x_n^{n-1} \end{aligned}$$

Demo: Newton-Cotes weight finder [cleared]

$$V^T \vec{w} = \vec{I}$$

Midpoint rule:

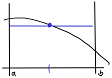
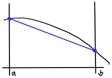
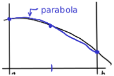




Turns out: It's general: quad rules w/ odd numbers of points have one "bonus" degree of exactness

Examples and Exactness

To what polynomial degree are the following rules exact?

Midpoint rule	$(b-a)f\left(\frac{a+b}{2}\right)$		1
Trapezoidal rule	$\frac{b-a}{2}(f(a)+f(b))$		1
Simpson's rule	$\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$		3

Interpolatory Quadrature: Accuracy

Let p_{n-1} be an interpolant of f at nodes $x_1, \ldots, x_n$ (of degree $n-1$)
Recall

$$\sum_i \omega_i f(x_i) = \int_a^b p_{n-1}(x)\mathrm{d}x.$$

What can you say about the accuracy of the method?

$$\left| \int_a^b f(x)\,dx - \int_a^b p_{n-1}(x)\,dx \right|$$

$$\leq \int_a^b |f(x) - p_{n-1}(x)|\,dx$$

$$\leq (b-a)\,\|f - p_{n-1}\|_\infty$$

$$\leq (b-a)\,C h^n \cdot \|f^{(n)}\|_\infty$$

$$= C h^{n+1} \|f^{(n)}\|_\infty$$

Quadrature: Overview of Rules

	n	Deg.	Ex.Int.Deg. (w/odd)	Intp.Ord.	Quad.Ord. (regular)	Quad.Ord. (w/odd)
		$n-1$	$(n-1)+1_{\text{odd}}$	n	$n+1$	$(n+1)+1_{\text{odd}}$
Midp.	1	0	1	1	2	3
Trapz.	2	1	1	2	3	3
Simps.	3	2	3	3	4	5
S. 3/8	4	3	3	4	5	5

- n: number of points
- "Deg.": Degree of polynomial used in interpolation ($= n - 1$)
- "Ex.Int.Deg.": Polynomials of up to (and including) this degree *actually* get integrated exactly. (including the odd-order bump)
- "Intp.Ord.": Order of Accuracy of Interpolation: $O(h^n)$
- "Quad.Ord. (regular)": Order of accuracy for quadrature predicted by the error result above: $O(h^{n+1})$
- "Quad.Ord. (w/odd):" Actual order of accuracy for quadrature given 'bonus' degrees for rules with odd point count

Observation: Quadrature gets (at least) 'one order higher' than interpolation–even more for odd-order rules. (i.e. more accurate)

Interpolatory Quadrature: Stability

Let p_n be an interpolant of f at nodes $x_1, \ldots, x_n$ (of degree $n-1$)

Recall

$$\sum_i \omega_i f(x_i) = \int_a^b p_n(x)\mathrm{d}x$$

What can you say about the stability of this method?

So, what quadrature weights make for bad stability bounds?