HW3 due tomorrow
Exam 1 starts in 10-15 days

Goals:
- matrix cond, 1/n.
- 2-norm (orthogonal mat, SVD review)
- solve (tri, Gauss, LU)
Matrix condition nr.

Solve $Ax = b$; matte $b = Ax$

$x$ residual

Relative error in $x$ $\leq \kappa \cdot$ rel. error in $b$

$\text{rel. error in } x \leq \kappa \cdot \text{rel. error in } b$

$5 - Ax = \hat{r}$

But actual error

"rel. error in $x" \leq \kappa \cdot \text{relative residual}$

\[ \frac{\|Ax - x\|}{\|x\|} \leq \kappa \cdot \frac{\|\hat{r}\|}{\|Ax\|} \leq \kappa \cdot \frac{\|x\|}{\|Ax\|} \leq \kappa \cdot \frac{\|x\|}{\|Ax\|} \]

Also, $\frac{\|Ax\|}{\|x\|} \leq \kappa$, \[
\frac{\|Ax\|}{\|x\|} \leq \kappa \cdot \frac{\|x\|}{\|Ax\|}
\]

\[ \kappa \cdot \text{error in } x \leq \kappa \cdot \text{rel. error in } A \]
Residual and Error: Relationship

How do the (norms of the) residual vector $r$ and the error $\Delta x = x - \hat{x}$ relate to one another?

(see above)
Changing the Matrix

So far, only discussed changing the RHS, i.e. \( Ax = b \rightarrow A\hat{x} = \hat{b} \).
The matrix consists of FP numbers, too—it, too, is approximate. I.e.

\[
Ax = \begin{bmatrix} b \end{bmatrix} \rightarrow \hat{A}\hat{x} = \begin{bmatrix} \hat{b} \end{bmatrix}.
\]

What can we say about the error due to an approximate matrix?

\[
\Delta\hat{x} = \hat{x} - \tilde{x} = A^{-1}A(\hat{x} - \tilde{x}) = A^{-1}(\hat{b} - A\tilde{x})
\]

\[
= A^{-1}(\hat{A}\tilde{x} - A\tilde{x})
\]

\[
= \Theta^{-1}(\hat{A} - A)\tilde{x}.
\]

\[
\|
\Delta\hat{x}\| = \|
A^{-1}\Delta A \hat{x}\| \leq \|
A^{-1}\|
\|
\Delta A\|
\|
\hat{x}\|
\]

\[
= \|
A^{-1}\|
\|
A\|
\|
\Delta A\|
\|
\hat{x}\|
\]

\[
= \kappa(A) \frac{\|
\Delta A\|}{\|
A\|} \|
\hat{x}\|
\]
\[
\frac{||A^2x||}{||x||} \leq \kappa(A) \frac{||DA||}{||A||}
\]

What if multiple things are wrong?

- Resulting error terms will be "error A", "error B"
  - Typically small

- Shrug, ignore
Changing Condition Numbers

Once we have a matrix $A$ in a linear system $Ax = b$, are we stuck with its condition number? Or could we improve it?

To rescale rows: left precondition
To rescale cols: right precondition

What is this called as a general concept?

Some matrix $M$: "left preconditioning" $(MA)x = MB \iff A\bar{x} = \bar{b}$

Hope: $\kappa(MA) < \kappa(A)$

Possible downside: $MA, M_b$ - need to be computed - may not be accurate
right precond $AMy = b \iff Ax = b$

$\tilde{x} = My$

1. solve for $\tilde{y}$ with $(AM)$ right precond sys matrix
2. recover $\tilde{x} = M\tilde{y}$. 
Recap: Orthogonal Matrices

What’s an orthogonal (=orthonormal) matrix?

One that satisfies $Q^T Q = I$ and $QQ^T = I$.

How do orthogonal matrices interact with the 2-norm?

$$
\|Qv\|_2^2 = (Qv)^T (Qv) = v^T Q^T Qv = v^T v = \|v\|_2^2.
$$
Singular Value Decomposition (SVD)

What is the Singular Value Decomposition of an $m \times n$ matrix?

$A = U \Sigma V^T$

- $U$ is $m \times m$ and orthogonal
- $V$ is $n \times n$ and orthogonal
- $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$ singular values
  $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$

\[\begin{pmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_n \end{pmatrix}\]

Economy SVD $\rightarrow$ Later

Existence, Computation $\rightarrow$ Later
Computing the 2-Norm

Using the SVD of $A$, identify the 2-norm.

Express the matrix condition number $\text{cond}_2(A)$ in terms of the SVD:
Not a matrix norm: Frobenius

The 2-norm is very costly to compute. Can we make something simpler?

What about its properties?
Frobenius Norm: Properties

Is the Frobenius norm induced by any vector norm?

How does it relate to the SVD?