HW3 due tomorrow

Exam 1 starts in 10-ish days

Goals:

- matrix cond nr.
  - 2-norm (orthogonal mat, SVD review)
- solve (tri, Gauss, LU)

Matrix cond. nr.

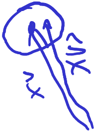
error prop.

solve $A\vec{x}=b$; rel. error in $\vec{x} \le \kappa \cdot$ rel. error in $\vec{b}$

mat-ve $b=A\vec{x}$; rel. error in $\vec{b} \le \kappa \cdot$ rel. error in $\vec{x}$

$\hat{x}$ $\qquad b - A\hat{x} = \vec{r}$ $\qquad$ bwd. error

residual: "rel. error in $\vec{x}$" $\le \kappa \cdot$ relative residual



rel. error small

$$\frac{\|\Delta \vec{x}\|}{\|\vec{x}\|} \le \kappa \cdot \frac{\|\vec{r}\|}{\|A\vec{x}\|}$$

computable

also true:

$$\frac{\|\Delta \vec{x}\|}{\|\vec{x}\|} \le \kappa \cdot \frac{\|\vec{r}\|}{\|A\vec{x}\|} \ ?$$

approx. matrix: "rel. error in $\vec{x}$" $\le \kappa \cdot$ rel. error in A.

Residual and Error: Relationship

How do the (norms of the) residual vector $\boldsymbol{r}$ and the error $\Delta \boldsymbol{x} = \boldsymbol{x} - \widehat{\boldsymbol{x}}$ relate to one another?

(see above)

Changing the Matrix

So far, only discussed changing the RHS, i.e. $A\boldsymbol{x} = \boldsymbol{b} \quad \to \quad A\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{b}}$.
The matrix consists of FP numbers, too—it, too, is approximate. I.e.

$$A\boldsymbol{x} = \boldsymbol{b} \quad \to \quad \widehat{A}\widehat{\boldsymbol{x}} = \boldsymbol{b}.$$

What can we say about the error due to an approximate matrix?

$$\Delta \vec{x} = \vec{x} - \hat{\vec{x}} = A^{-1}A(\vec{x} - \hat{\vec{x}}) = A^{-1}(\vec{b} - A\hat{\vec{x}})$$

$$= A^{-1}(\hat{A}\hat{\vec{x}} - A\hat{\vec{x}})$$

$$= A^{-1}(\hat{A} - A)\hat{\vec{x}}$$

$$\|\Delta\vec{x}\| = \|A^{-1}\Delta A\,\hat{\vec{x}}\| \le \|A^{-1}\|\,\|\Delta A\|\,\|\hat{\vec{x}}\|$$

$$= \|A^{-1}\|\,\|A\|\,\frac{\|\Delta A\|}{\|A\|}\,\|\hat{\vec{x}}\|$$

$$= \kappa(A)\,\frac{\|\Delta A\|}{\|A\|}\,\|\hat{\vec{x}}\|$$

$$\frac{\|\Delta \vec{x}\|}{\|\hat{x}\|} \leq \kappa(A) \frac{\|\Delta A\|}{\|A\|}$$

What if multiple things are wrong?

→ Resulting error terms will be

"error A", "error B"

→ typically small

→ shrug, ignore

Changing Condition Numbers

Once we have a matrix A in a linear system $A\boldsymbol{x} = \boldsymbol{b}$, are we stuck with its condition number? Or could we improve it?

to rescale rows: left precond

to rescale cols: right precond.

↳ if κ matters, always start by rescaling

What is this called as a general concept?

Some matrix invertible M:

"left preconditioning": $(MA)\vec{x} = M\vec{b} \iff A\vec{x} = \vec{b}$

Hope: $\kappa(MA) < \kappa(A)$

possible downside: MA, Mb
- need to be computed
- may be inaccurate.

"right precond." $(AM)\vec{y} = \vec{b} \iff A\vec{x} = \vec{b}$

$$\vec{x} = M\vec{y}$$

① solve for $\vec{y}$ with (AM) right precond sys. matrix

② recover $\vec{x} = M\vec{y}$.

Recap: Orthogonal Matrices

What's an *orthogonal (=orthonormal) matrix*? Q

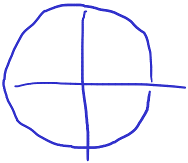
One that satisfies $Q^T Q = I$ and $QQ^T = I$. $\Rightarrow \quad Q^T = Q^{-1}$

How do orthogonal matrices interact with the 2-norm?

$\|\cdot\| \geq 0$

$\|Q\vec{v}\|_2 = \|\vec{v}\|_2$

$$\|Q\mathbf{v}\|_2^2 = (Q\mathbf{v})^T(Q\mathbf{v}) = \mathbf{v}^T Q^T Q\mathbf{v} = \mathbf{v}^T\mathbf{v} = \|\mathbf{v}\|_2^2.$$



$\|\vec{x}\|_2^2 = \vec{x}^T\vec{x}$

Euclidean length

"Rotation and Flipping don't change

Singular Value Decomposition (SVD)

What is the *Singular Value Decomposition* of an $m \times n$ matrix?

$$A = U \Sigma V^T$$

"Full"

- U is $m \times m$ and orthogonal — col of U: "left sing. vec."
- V is $n \times n$ and orthogonal — col of V: "right sing vec."
- $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$ singular values, $\Sigma \in \mathbb{R}^{m \times n}$

$\sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_n \ge 0$

"economy SVD" → later

existence, computation: later

Computing the 2-Norm

$A = U\Sigma V^+$

Using the SVD of A, identify the 2-norm. matrix

A $\quad \|A\|_2$

~~wishful thinking~~

$$\|A\|_2 = \|U\Sigma V^+\|_2 \overset{?}{=} \|\Sigma\|_2 = \max_{\|x\|_2=1} \|\Sigma \tilde{x}\|_2 = \sigma_1$$

diagonal

"exercise"

$$\|QA\|_2 = \max_{\|x\|_2=1} \|QAx\|_2 = \max_{\|x\|_2=1} \|A\tilde{x}\|_2 = \|A\|_2$$

$$\|AQ\|_2 = \ldots = \|A\|_2$$

Express the matrix condition number $\text{cond}_2(A)$ in terms of the SVD:

$$\kappa_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2 = \sigma_1/\sigma_n$$

$$A^{-1} = (U\Sigma V^T)^{-1} = V^{-T}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^+$$

$$\|A^{-1}\|_2 = \frac{1}{\sigma_n}$$

if A is not invertible, $\sigma_n = 0 \Rightarrow \kappa_2(A) = \sigma_1/0 = \infty$

Not a matrix norm: Frobenius

The 2-norm is very costly to compute. Can we make something simpler?

What about its properties?

Frobenius Norm: Properties

Is the Frobenius norm induced by any vector norm?

How does it relate to the SVD?