Exam 1: content thru end lec 8

HW4 (short to allow exam prep)

Goals:

- Frobenius
- ~~Solv~~ Solve

$\|A\|_2 = \sigma_{max}$

SVD? expensive

$\kappa(A) = \|A\| \, \|A^{-1}\| < 1?$

$\geq \|AA^{-1}\| = \|I\| = 1$

$\|AB\| \leq \|A\| \, \|B\|$

Computing the 2-Norm

Using the SVD of A, identify the 2-norm.

σ_{max}

Express the matrix condition number $\operatorname{cond}_2(A)$ in terms of the SVD:

σ_1 / σ_n

Not a matrix norm: Frobenius

The 2-norm is very costly to compute. Can we make something simpler?

$$\|A\|_F := \sqrt{\sum_{i,j} |A_{ij}|^2}$$

What about its properties?

- definiteness
- $\|\alpha A\|_F = |\alpha| \, \|A\|_F$
- triangle ineq.
- $\|AB\|_F \le \|A\|_F \, \|B\|_F$ (Cauchy-Schwarz)

Frobenius Norm: Properties

$\|A\|_* = \max_{\|x\|_*=1} \|Ax\|_*$

Is the Frobenius norm induced by any vector norm?

$\|I\|_F = n$ for $I \in \mathbb{R}^{n\times n}$ $\rightarrow$ no, for induced n: $\|I\| = 1$

How does it relate to the SVD?

$A = U\Sigma V^T$

$\|A\|_F = \|\Sigma\|_F = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$

$\|QA\|_2 = \|A\|_2$ $\quad \|QA\|_F = \|A\|_F$

la.norm(A) → Fro

(A, "Fro")

Solving Systems: Simple cases

$3n^5 - 15n^2 + \ldots \Rightarrow O(n^5)$

Solve $D\boldsymbol{x} = \boldsymbol{b}$ if D is diagonal. (Computational cost?)

$x_i = b_i / d_{ii}$ $\quad O(n) \quad$ as $n \to \infty$

Solve $Q\boldsymbol{x} = \boldsymbol{b}$ if Q is orthogonal. (Computational cost?) $\quad Q \in \mathbb{R}^{n \times n}$

$Q\vec{x} = \vec{b} \mid Q^T \cdot \quad \cancel{Q^T Q}\,\vec{x} = Q^T\vec{b} \quad O(n^2)$

Given SVD $A = U\Sigma V^T$, solve $A\boldsymbol{x} = \boldsymbol{b}$. (Computational cost?)

$U\Sigma V^T\vec{x} = \vec{b} \mid U^T \cdot$

$\Leftrightarrow \Sigma \underbrace{V^T x}_{\vec{y}} = U^T b$

$\Leftrightarrow \Sigma\vec{y} = U^T b$

$V^T\vec{x} = \vec{y}$

$\Leftrightarrow \vec{x} = V\vec{y}$

$O(n^2)$

$$(A\vec{x})_i = \sum_j A_{i,j} x_j \qquad i \in \{1 \dots n\}$$

n multiplications

n-1 additions

n outputs

$$n(n+n-1)$$

$$n(2n-1) = O(n^2)$$

Solving Systems: Triangular matrices

Solve

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

$a_{33} \cdot z + a_{34} \cdot w = b_3$

$a_{44} \cdot w = b_4$ ✓

Demo: Coding back-substitution [cleared]

What about non-triangular matrices?

Gaussian elimination

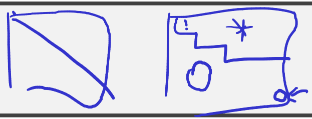
Gaussian Elimination

Demo: Vanilla Gaussian Elimination [cleared]

What do we get by doing Gaussian Elimination?

REF

How is that different from being upper triangular?



What if we do not just eliminate downward but also upward?

Gauss-Jordan → won't consider

If inv: GE gives ∇ : can use BS

LU Factorization

What is the LU factorization?

from GE

$A = LU$

$\exists$? $\to$ later

Solving $A\boldsymbol{x} = \boldsymbol{b}$

Does LU help solve $A\boldsymbol{x} = \boldsymbol{b}$?

$$L\underbrace{U\vec{x}}_{\vec{y}} = b$$

solve $L\vec{y} = b$ for y (using FS)

solve $Ux = \vec{y}$ for $\vec{x}$ (using BS)

Determining an LU factorization

AB

$$A = \begin{pmatrix} a_{11} & \vec{a}_{12}^T \\ \vec{a}_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 \\ \vec{l}_{21} & L_{22} \end{pmatrix} \begin{pmatrix} u_{11} & \vec{u}_{12}^T \\ 0 & U_{22} \end{pmatrix}$$

$$\begin{pmatrix} u_{11} & \vec{u}_{12}^T \\ 0 & U_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ \vec{l}_{21} & L_{22} \end{pmatrix} \quad \begin{pmatrix} a_{11} & \vec{a}_{12}^T \\ \vec{a}_{21} & A_{22} \end{pmatrix}$$

$a_{11} = l_{11} \cdot u_{11}$ ← impose $l_{ii} = 1$

↪ $u_{11} = a_{11}$

$\vec{a}_{21} = \vec{l}_{21} \cdot u_{11}$

↪ $\vec{l}_{21} = \vec{a}_{21} / u_{11}$

$\vec{a}_{12}^T = \vec{u}_{12}^T$

↪ $\vec{u}_{12} = \vec{a}_{12}$

$A_{22} = \vec{l}_{21} \cdot \vec{u}_{12}^T + L_{22} U_{22}$

Outer prod!

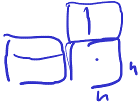
Demo: LU Factorization [cleared]

$$L_{22} U_{22} = A_{22} - \vec{l}_{21} \vec{u}_{12}^{\,T}$$ ✓ RHS

$\Uparrow$

"recurse" to LU fact of size $(n-1) \times (n-1)$

Computational Cost



What is the computational cost of multiplying two $n \times n$ matrices?

$$O(n^3)$$

- $u_{11} = a_{11}$, $\boldsymbol{u}_{12}^T = \boldsymbol{a}_{12}^T$. $\leftarrow$ $O(n)$
- $\ell_{21} = \boldsymbol{a}_{21}/u_{11}$. $\leftarrow$ $O(n)$
- $L_{22}U_{22} = A_{22} - \ell_{21}\boldsymbol{u}_{12}^T$. $\leftarrow$ $O(n^2)$

n times

What is the computational cost of carrying out LU factorization on an $n \times n$ matrix?

$$O(n^3)$$

Demo: Complexity of Mat-Mat multiplication and LU [cleared]

LU: Failure Cases?

Is LU/Gaussian Elimination bulletproof?