: 30

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underbrace{\vec{a}_{21}}_{\text{all zeros}} = \underbrace{u_{11}}_{0} \; \underbrace{\vec{l}_{21}}_{\text{anything, maybe } 0}$$

cond: $\dfrac{\|\vec{y} - \hat{\vec{y}}\|}{\|\vec{y}\|} \leq \kappa \dfrac{\|\vec{x} - \hat{\vec{x}}\|}{\|\vec{x}\|}$

bw error

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr] & y \\ \tilde{x} \arrow[r, "f"'] \arrow[ur, "f"] & \tilde{y} \end{tikzcd}$$

To bound fw error: $\|x - \tilde{x}\| / \|x\|$

- bound bw error
- use cond bound

f "true answer" | $\tilde{f}$ "method"

cond. | accuracy

(bw) stability

abs error $|y-\tilde{y}|$

rel. error $\frac{|y-\tilde{y}|}{|y|}$

f g

κ_f κ_g

$\kappa(f \circ g)$ $\kappa_f \kappa_g \frac{\|Ax\|}{\|A\|\|x\|}$

$(f \circ g)(x) = f(g(x))$

$$\kappa(AB) \le \kappa(A)\,\kappa(B)$$

$$= \|AB\|\,\|(AB)^{-1}\| = \|AB\|\,\|B^{-1}A^{-1}\|$$

$$\le \|A\|\,\|B\|\,\|B^{-1}\|\,\|A^{-1}\|$$

$$= \kappa(A)\,\kappa(B)$$

$$1 = \kappa(I) \le \kappa(A)\cdot\kappa(A^{-1})$$

$$\pi\cdot 10^{100} + \cancel{\frac{\pi}{2}}$$

---

subn with acc:

nom 1|

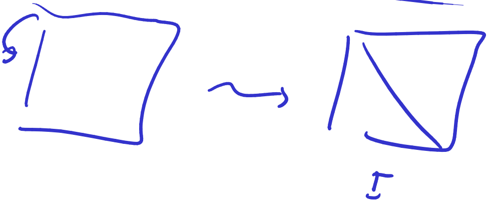
subn 0.

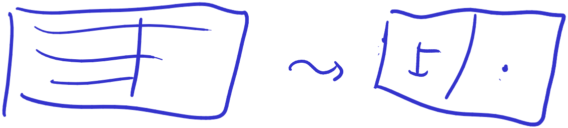
norm $(1.\underbrace{000 \quad\quad\quad 0}_{})_2 \, 2^{-1022} \Big/ 2 =$

$(0.1 \quad\quad\quad\quad\quad\quad) \, 2^{-1022}$

stored as $-1023+1023=0$

Obj:





asymp: $O(n^3)$

LU on Blocks: The Schur Complement

Given a linear system

$$\left[\begin{array}{cc|c} A & B & \boldsymbol{b}_1 \\ C & D & \boldsymbol{b}_2 \end{array}\right], \quad -CA^{-1}\cdot \quad \oplus \quad \cancel{-A^{-1}C\cdot}$$

can we do 'block Gaussian elimination' to get a *block triangular matrix*?

"Schur complement"

$$(-CA^{-1})A = -C$$

$$\left[\begin{array}{cc|c} A & B & \vec{b}_1 \\ 0 & D-CA^{-1}B & \vec{b}_2 - CA^{-1}\vec{b}_1 \end{array}\right]$$

LU: Special cases

$PA = LU$

What happens if we feed a non-invertible matrix to LU?

$P\,A = L\,U$

L: diag of ones

not inv

inv $A = LU$

A: $m \times n$ $\quad$ L: $m \times k$ $\quad$ U: $k \times n$?

What happens if we feed LU an $m \times n$ non-square matrices?

$Ax = b \rightsquigarrow A$ square

tall + skinny $\qquad$ short + fat

$m>n \quad m \times n \quad n \times n \qquad\qquad m<n \quad m \times m \quad m \times n$

"reduced" LU $\qquad\qquad$ "economy" LU

Round-off Error in LU without Pivoting

Consider factorization of $\begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$ where $\epsilon < \epsilon_{\text{mach}}$:

Round-off Error in LU with Pivoting

Permuting the rows of A in partial pivoting gives $PA = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix}$

Changing matrices

Seen: LU cheap to re-solve if RHS changes. (Able to keep the expensive bit, the LU factorization) What if the *matrix* changes?

$A \rightsquigarrow PA = LU$ $\tilde{A} = A + \underbrace{\vec{u}\vec{v}^T}$ ← rank-1 update of A

solve $\hat{A}x = b$ → how?

Sherman-Morrison formula:

$$\tilde{A}^{-1} = (A + \vec{u}\vec{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\vec{u}\vec{v}^T A^{-1}}{1 + \vec{v}^T A^{-1}\vec{u}}$$

Demo: Sherman-Morrison [cleared]

v $\iff \vec{x} = \hat{A}^{-1}\vec{b}$

$$= A^{-1}\vec{b} - \frac{A^{-1}\vec{u}\,\vec{v}^{T}A^{-1}}{1+\vec{v}^{T}A^{-1}\vec{u}}\,\vec{b}$$

(expensive) $$= A^{-1}\vec{b} - \frac{(A^{-1}(\vec{u}\,\vec{v}^{T}))A^{-1}}{1+\vec{v}^{T}A^{-1}\vec{u}}\,\vec{b}$$

cn

(cheaper) $$= \underbrace{(A^{-1}\vec{b})}_{cn} - \frac{(A^{-1}(\vec{u}(\vec{v}^{T}(A^{-1}\vec{b}))))}{1+\vec{v}^{T}\underbrace{(A^{-1}\vec{u})}_{cn}}$$

Solve on $\hat{A} = A + uv^{T}$ at $O(n^2)$ cost.

In-Class Activity: LU

In-class activity: LU and Cost