Today

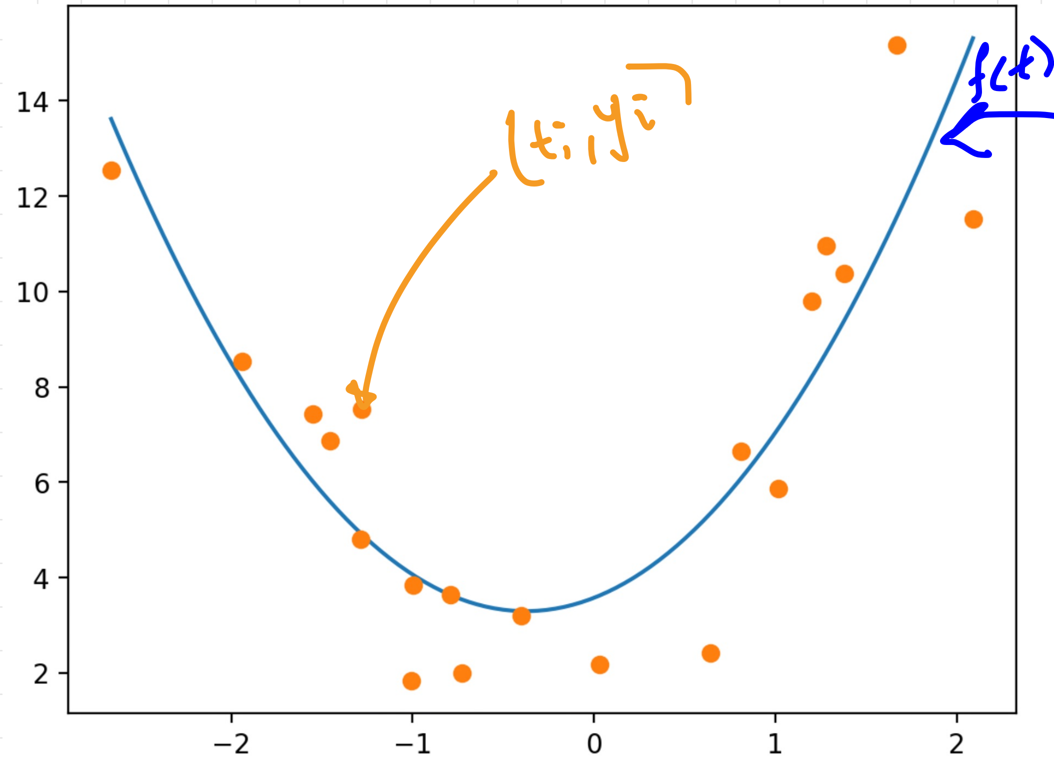
① Finish up LU : round-off
slides # 83/84

② Linear Least Squares

① Think about small ε

$$A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix}$$

② linear least squares:



We have data (t_i, y_i)

We want a function

$$f(t) = x_1 + x_2 t + x_3 t^2$$

so that

$$y_i = f(t_i) = x_1 + x_2 t_i + x_3 t_i^2$$

$$y_i = f(t_i) = x_1 + x_2 t_i + x_3 t_i^2$$

$\downarrow$

$$y_1 = 1\,x_1 + x_2 t_1 + x_3 t_1^2$$

$$y_2 = x_1 + x_2 t_2 + x_3 t_2^2$$

$$\vdots$$

$$y_m = x_1 + x_2 t_m + x_3 t_m^2$$

$$\rightarrow \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ & \vdots & \\ 1 & t_m & t_m^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$\underset{m \times 3}{} \qquad \underset{3 \times 1}{} \qquad \underset{m \times 1}{}$$

$m \times 3$ $\quad$ 3×1 $\quad$ $m \times 1$

OK so we have: Find $\underline{x}$ such that

$$A\underline{x} = b$$

$m \times n \quad n \times 1 \quad m \times 1$

Q: Does this have a solution? $m = n$ ✓ yes

$m > n$ ✗ no, only if

$b \in \text{col span}(A)$

Different question: Find $\underline{x}$ such that

$$\|Ax - b\|_2 \text{ is minimized}$$

Q: solution exist? ✓ yes

Q: unique? ✗ yes, if A is full rank

How do we find $\underline{x}$?

s.t.

$$\underline{x} \leftarrow \min_{\underline{x}} \|Ax - b\|_2$$

Two main approaches:

1) direct minimization

2) transform A

① let $f(x) = \|Ax - b\|_2^2$

$$= \|b - Ax\|_2^2$$

$$= (b - Ax)^T (b - Ax)$$

$(b^T - x^T A^T)$

$$= b^T b - x^T A^T b - b^T A x + x^T A^T A x$$

$$= b^T b - 2\, x^T A^T b + x^T A^T A x$$

$\|v\|_2^2 = v^T v$

→ find $\nabla_x f$

→ set $\nabla_x f = 0$

→ find x

Two ways to find $\nabla f(x)$

1) write all components out with $\sum_{i,j}$

2) $\lim_{\tau \to 0} \dfrac{f(x + \tau \cdot z) - f(x)}{\tau}$

(try it!)

$$f(\underline{y}) = b^T b - 2x^T A^T b + x^T A^T A x$$

$$\rightarrow \nabla f(\underline{x}) = 0 - 2A^T b + 2A^T A x = 0$$

set = 0

$\rightarrow$

$$2A^T A x = 2A^T b$$

$\rightarrow$

$$A^T A x = A^T b$$

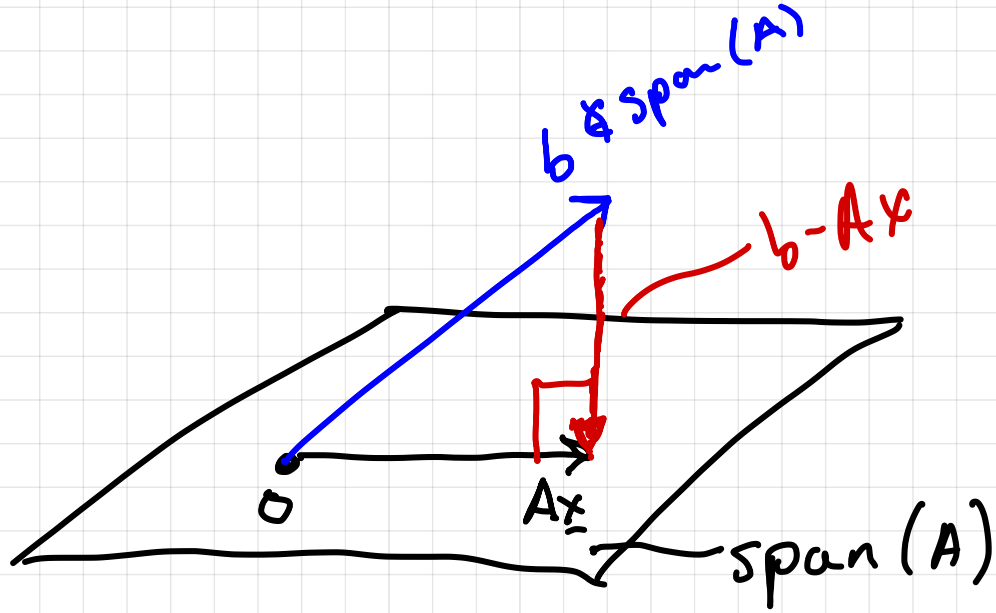
The Normal Equations

$$\underbrace{n \times m \quad m \times n}_{n \times n} \cdot n \times 1 = n \times m \quad m \times 1$$

$$\cdot n \times 1 = n \times 1$$

$$\bar{A}\,\underline{x} = \hat{b}$$

②



$$Ax = b$$

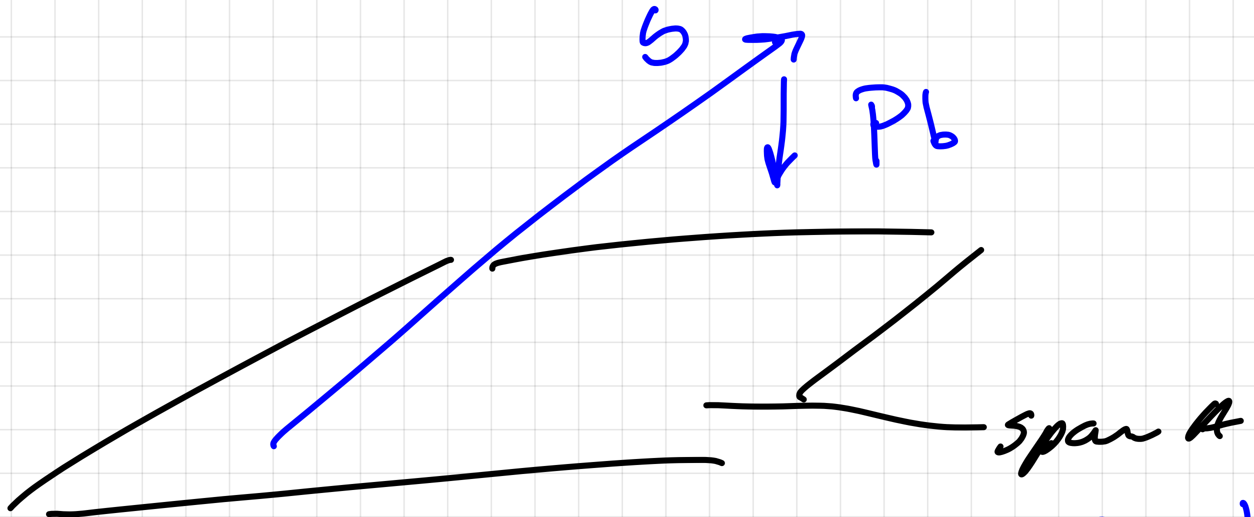
$$\min \|Ax - b\|_2 \;?$$

$$\underbrace{\begin{bmatrix} 1 & t_1 & t_2 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightsquigarrow x_1 \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} t_1 \\ \vdots \\ t_m \end{bmatrix} + x_3 \begin{bmatrix} t_1^2 \\ \vdots \\ t_m^2 \end{bmatrix}$$

$\to$ $b - Ax \perp$ span A

$\to$ $a^T (b - Ax) = 0$ for all $a = col(A)$

$\to$ $A^T (b - Ax) = 0 \longrightarrow A^T A x = A^T b$



We want a projection of b

$P^2 = P$

$P = P^T$

know $A^TAx = A^Tb$

n×n m×m

→ $x = \underbrace{(A^TA)^{-1}A^T}_{A^+} b$

pseudoinverse (page 113)

→ $Ax = \underbrace{A(A^TA)^{-1}A^T}_{P} b$

$$P^2 = A(A^TA)^{-1}\underbrace{A^T A (A^TA)^{-1}}_{I} A^T$$

$$= A(A^TA)^{-1}A^T$$

$$= P$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \underline{x} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} -15 \\ 4 \end{bmatrix}$$

$$10 \quad \underline{x} = 11$$

$$\Rightarrow \quad x = 11/10$$

$$A^T A x = A^T b$$

$$\downarrow$$

$$x \longrightarrow \min \|Ax - b\|_2$$

$$\not\downarrow$$

$$Ax = b$$

$$y = 1 \cdot x + b \cdot x + a^2 \cdot x$$

$$y = \overset{f}{1} \cdot x + b \cdot x + d \cdot x$$

$$a = \sqrt{d}$$

$\boxed{\text{p107}}$