Exam 1

Jupyter like breakage $\Theta$ 1 pt.

Look at exam in office hours

HW hints $\rightarrow$ hw 6 pl

$A = UEV^T$

$m \times n, m > n$

$A \tilde{x} \approx \tilde{b}$

For any ONB: $\tilde{w}_1, \ldots, \tilde{w}_n$

$\tilde{v} = \sum_{i=n+1}^{m} (r_i u_i) \tilde{w}_i$

$\tilde{v} \in \text{span} \{ \tilde{w}_1, \ldots, \tilde{w}_n \}$
\[
\| Ax - b \|_1 \\
\leq \| Q^T (Qx - b) \|_1 = \| x - Q^T b \|_1
\]
Computing QR

- Gram-Schmidt
- Householder Reflectors
- Givens Rotations

**Demo:** Gram-Schmidt–The Movie [cleared] (shows modified G-S)
**Demo:** Gram-Schmidt and Modified Gram-Schmidt [cleared]
**Demo:** Keeping track of coefficients in Gram-Schmidt [cleared]

Seen: Even modified Gram-Schmidt still unsatisfactory in finite precision arithmetic because of roundoff.

**NOTE:** Textbook makes further modification to ‘modified’ Gram-Schmidt:
- Orthogonalize *subsequent* rather than *preceding* vectors.
- Numerically: no difference, but sometimes algorithmically helpful.
Economical/Reduced QR

Is QR with square $Q$ for $A \in \mathbb{R}^{m \times n}$ with $m > n$ efficient?

"economy QR":

\[
\begin{align*}
A &\rightarrow Q &\rightarrow R \\
\end{align*}
\]

\[
\text{\hat{\mathbf{x}}} = \mathbf{R}_{\text{top}}^{-1} \left( \mathbf{Q}^T \mathbf{b} \right)_{\text{top}}
\]

\[\text{\hat{\mathbf{x}}} \text{ still works, even with economy}\]
Is QR with square $Q$ for $A \in \mathbb{R}^{m \times n}$ with $m > n$ efficient?

No. Can obtain economical or reduced QR with $Q \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{n \times n}$. Least squares solution process works unmodified with the economical form, though the equivalence proof relies on the ‘full’ form.
Constructing Reflections

Given a plane represented by its (unit) normal vector \( \mathbf{n} \), construct a reflection about that plane.

\[
\mathbf{x}_{\text{ref}} = \mathbf{x} - 2 (\mathbf{x}^T \mathbf{n}) \mathbf{n}
\]

\[
\mathbf{n} = I - 2 \mathbf{n} \mathbf{n}^T
\]

\[
\mathbf{r} \mathbf{x} = \mathbf{x} - 2 \mathbf{n} (\mathbf{n}^T \mathbf{x})
\]
Householder Transformations

Find an *orthogonal* matrix $Q$ to zero out the lower part of a vector $a$.  

Let $\hat{a} = \text{first col of } A$

Want: reflector $H$, solve $H \hat{a} = \begin{pmatrix} x_1 \\ \vdots \end{pmatrix}$

$$\hat{a} = a - \|a\| e_1$$

$$n = \frac{a - \|a\| e_1}{\|a - \|a\| e_1\|}$$

$$H = I - 2 \hat{n} \hat{n}^T = I - 2 \frac{\hat{n} \hat{n}^T}{\hat{n}^T \hat{n}}$$

Cost of $Hx$?

$Iz : O(n)$

$2\hat{n}^T x : O(n)$
Householder Reflectors: Properties

Seen from picture (and easy to see with algebra):

\[ H\mathbf{a} = \pm \|\mathbf{a}\|_2 \mathbf{e}_1. \]

Remarks:

▶ Q: What if we want to zero out only the \( i + 1 \)th through \( n \)th entry?
  A: Use \( \mathbf{e}_i \) above.

▶ A product \( H_n \cdots H_1 \mathbf{A} = \mathbf{R} \) of Householders makes it easy (and quite efficient!) to build a QR factorization.

▶ It turns out \( \mathbf{b} = \mathbf{a} + \|\mathbf{a}\|_2 \mathbf{e}_1 \) works out, too—just pick whichever one causes less cancellation.

▶ \( H \) is symmetric

▶ \( H \) is orthogonal

Demo: 3x3 Householder demo [cleared]
$$\begin{pmatrix}
\begin{array}{c}
\mathbf{x} \\
\vdots \\
\mathbf{x}
\end{array}
\end{pmatrix}$$

$$\mathbf{s}_n^2 = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}$$

$$\mathbf{v} = \mathbf{a} - \mathbf{a}_n \| \mathbf{e}_i \| = \begin{pmatrix}
0 \\
\vdots \\
x
\end{pmatrix}$$

$$\Lambda = I - \frac{\mathbf{v} \mathbf{v}^T}{\mathbf{v}^T \mathbf{v}}$$
Givens Rotations

If reflections work, can we make rotations work, too?

\[
\begin{pmatrix}
  c & s \\
  -s & c
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  a_2
\end{pmatrix} = \begin{pmatrix}
  \pm \sqrt{a_1^2 + a_2^2} \\
  0
\end{pmatrix}
\]

\[
c = a_1 / \|a\|_2 \\
\|a\|_2 = a_2 / \|a\|_2
\]

**Demo:** 3x3 Givens demo [cleared]
\[
\begin{bmatrix}
G_3 & G_2 & G_1
\end{bmatrix}
Q^T =
\begin{bmatrix}
x & x & x & x & x
\end{bmatrix}
\Rightarrow
Q^T A \in \mathbb{R}
\]

\[
G_1 = \begin{pmatrix}
1 & (c_3 s_1) \\
0 & (c_3 s_1)
\end{pmatrix}
\]

\[
G_2 = \begin{pmatrix}
c_1 & s_1 \\
-s_1 & c_1
\end{pmatrix}
\]

\[
G_3 = \begin{pmatrix}
1 & (c_3 s_1) \\
0 & (c_3 s_1)
\end{pmatrix}
\]
Rank-Deficient Matrices and QR

What happens with QR for rank-deficient matrices?