Exam 1

Jupyter lite breakage ⊕ 1 pt.

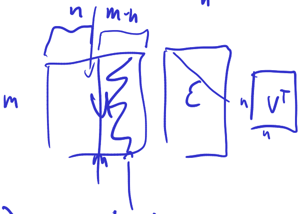
look at exam in office hours

hw hints → hw 6 p1

$\delta\vec{b} = \varepsilon \vec{u}_n$

$A = U \Sigma V^T$

$m \times n \quad m > n$



$A\vec{x} \cong \vec{b}$

$\vec{r} \in \mathrm{span}(\quad)$

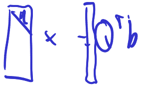
$\vec{r} = \sum_{i=n+1}^{m} (r^T u_i)\, \vec{u}_i$

For any ONB: $\vec{w}_1, \dots, \vec{w}_n$

$$\vec{z} = \sum_i \underbrace{(z^T u_i)}_{?}\, \vec{w}_i$$

$$\|Ax-b\|_2$$

$$= \|Q^T(QRx-b)\|_2 = \|Rx - Q^Tb\|_2$$



Computing QR

- Gram-Schmidt
- Householder Reflectors
- Givens Rotations

Demo: Gram-Schmidt–The Movie [cleared] (shows *modified G-S*)
Demo: Gram-Schmidt and Modified Gram-Schmidt [cleared]
Demo: Keeping track of coefficients in Gram-Schmidt [cleared]
Seen: Even modified Gram-Schmidt still unsatisfactory in finite precision arithmetic because of roundoff.

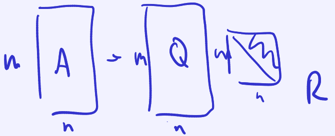
NOTE: Textbook makes further modification to 'modified' Gram-Schmidt:

- Orthogonalize *subsequent* rather than *preceding* vectors.
- Numerically: no difference, but sometimes algorithmically helpful.

Economical/Reduced QR

Is QR with square Q for $A \in \mathbb{R}^{m\times n}$ with $m > n$ efficient?

"economy QR":



$$\tilde{x} = R_{top}^{-1}\,(Q^T b)_{top}$$

still works, even with economy

Economical/Reduced QR

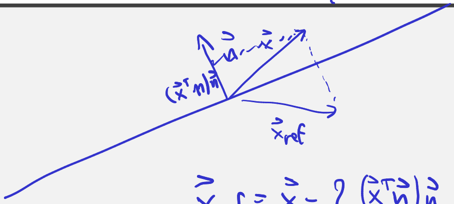
Is QR with square Q for $A \in \mathbb{R}^{m \times n}$ with $m > n$ efficient?

No. Can obtain economical or reduced QR with $Q \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{n \times n}$. Least squares solution process works unmodified with the economical form, though the equivalence proof relies on the 'full' form.

Constructing Reflections

Given a plane represented by its (unit) normal vector $\boldsymbol{n}$, construct a reflection about that plane.

$\|\vec{n}\|_2 = 1$



$$\vec{x}_{ref} = \vec{x} - 2(\vec{x}^T \vec{n})\vec{n}$$

$$\vec{x}_{ref} = R\,\vec{x}$$

$$R = I - 2\vec{n}\,\vec{n}^T$$

reflection

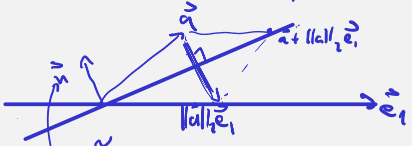
$$R\vec{x} = \vec{x} - 2\vec{n}\,(\vec{n}^T \vec{x})$$

Householder Transformations

Find an *orthogonal* matrix Q to zero out the lower part of a vector $\boldsymbol{a}$.

let $\vec a$ = first col of A

Want: reflector H_1 so that $H_1 \vec a = \alpha \vec e_1 = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← first col of R



$\tilde{\vec n} = \vec a - \|\vec a\| \vec e_1$

$$\vec n = \frac{\tilde{\vec n}}{\|\tilde{\vec n}\|}$$

$$H = I - 2\vec n \vec n^T = I - 2\frac{\tilde{\vec n}\tilde{\vec n}^T}{\tilde{\vec n}^T \tilde{\vec n}}$$

cost of $H\vec x$?

$I\vec x$: $O(n)$

$2\vec n(\vec n^T x)$: $O(n)$

Householder Reflectors: Properties

Seen from picture (and easy to see with algebra):

$$H\boldsymbol{a} = \pm \|\boldsymbol{a}\|_2 \, \boldsymbol{e}_1.$$

Remarks:

- Q: What if we want to zero out only the $i+1$th through nth entry?
  A: Use $\boldsymbol{e}_i$ above.
- A product $H_n \cdots H_1 A = R$ of Householders makes it easy (and quite efficient!) to build a QR factorization.
- It turns out $\cancel{\boldsymbol{v}}$ $\tilde{\boldsymbol{v}}$ $= \boldsymbol{a} + \|\boldsymbol{a}\|_2 \, \boldsymbol{e}_1$ works out, too–just pick whichever one causes less cancellation.

  $\tilde{v} = \vec{a} - \|a_2\| \vec{e}_1$
- H is symmetric
- H is orthogonal $H^T H$ ..

$$\pm \begin{pmatrix} \|a\| \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

choose so no subtraction happens

Demo: 3x3 Householder demo [cleared]

$$\begin{pmatrix} * & \boxed{x} & x \\ 0 & \boxed{x} & x \\ 0 & \boxed{x} & x \end{pmatrix}$$

$$\tilde{\vec{a}} = \begin{bmatrix} 0 \\ x \\ x \end{bmatrix}$$

$$\vec{v} = \vec{a} - \|a_2\| e_1 = \begin{pmatrix} 0 \\ x' \\ x' \end{pmatrix}$$

$$H = I - \frac{vv^T}{v^Tv}$$

Givens Rotations

If reflections work, can we make rotations work, too?

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \underbrace{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{\vec{a}} = \begin{bmatrix} \pm\sqrt{a_1^2 + a_2^2} \\ 0 \end{bmatrix}$$

$$c = a_1 / \|\vec{a}\|_2$$
$$s = a_2 / \|\vec{a}\|_2$$

Demo: 3x3 Givens demo [cleared]

$$\underbrace{G_3 G_2 G_1}_{Q^T} \begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} = \begin{pmatrix} \circ & \circ & \circ \\ & \circ & \circ \\ & & \circ \end{pmatrix}$$

$$Q^T A = R$$

$$G_1 = \begin{pmatrix} 1 & & \\ & c_1 & s_1 \\ & -s_1 & c_1 \end{pmatrix}$$

$$G_2 = \begin{pmatrix} c_2 & s_2 & \\ -s_2 & c_2 & \\ & & 1 \end{pmatrix}$$

$$G_3 = \begin{pmatrix} 1 & & \\ & c_3 & s_3 \\ & -s_3 & c_3 \end{pmatrix}$$

Rank-Deficient Matrices and QR

What happens with QR for rank-deficient matrices?