- Feedbach
- Exam 2: starts next Friday
- Goals :
 - eigenvalues - linalg review, motivation - sons itivity - methods

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix. $\mathbf{x} \neq 0$ is called an *eigenvector* of A if there exists a λ so that $A\mathbf{x} = \lambda \mathbf{x}.$ \Rightarrow $A(\mathbf{x}) - \lambda(\mathbf{x})$

- ln that case, λ is called an *eigenvalue*.
- The set of all eigenvalues $\lambda(A)$ is called the *spectrum*.
- ▶ The *spectral radius* is the magnitude of the biggest eigenvalue:

$$ho(A) = \max\left\{ |\lambda| : \lambda(A)
ight\}$$

Finding Eigenvalues

How do you find eigenvalues?

 $A\mathbf{x} = \lambda \mathbf{x} \Leftrightarrow (A - \lambda I)\mathbf{x} = 0$ $\Leftrightarrow A - \lambda I \text{ singular} \Leftrightarrow \det(A - \lambda I) = 0$

det $(A - \lambda I)$ is called the *characteristic polynomial*, which has degree n, and therefore n (potentially complex) roots.

Does that help algorithmically? Abel-Ruffini theorem: for $n \ge 5$ is no general formula for roots of polynomial. IOW: no.

- ▶ For LU and QR, we obtain *exact* answers (except rounding).
- For eigenvalue problems: not possible—must *iterate*.

Demo: Rounding in characteristic polynomial using SymPy [cleared]

Multiplicity

What is the *multiplicity* of an eigenvalue?

Actually, there are two notions called multiplicity:

 Algebraic Multiplicity: multiplicity of the root of the characteristic polynomial

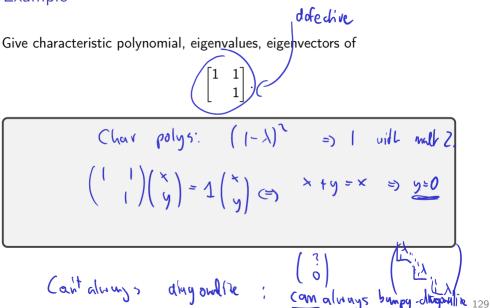
p= (1-1)3

Geometric Multiplicity: #of lin. indep. eigenvectors

In general: $AM \ge GM$.

If AM > GM, the matrix is called *defective*.

An Example



Diagonalizability
When is a matrix called diagonalizable? If not defective

$$deg ((P) = \int_{-\infty}^{An-GM} \exists n lin indep eigen vectors, (across all eign vectors, (across all eign vectors))
\times - (V_1 = V_1) \leftarrow unt rix of eign vectors)
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Similar Matrices

Related definition: Two matrices A and B are called similar if there exists an invertible matrix X so that $A = XBX^{-1}$.

In that sense: "Diagonalizable" = "Similar to a diagonal matrix".

Observe: Similar A and B have same eigenvalues. (Why?)

Suppose
$$A_{x}^{2} = \lambda \tilde{\chi}$$
 $(\tilde{\chi}^{2} + \tilde{\partial})$, $B^{2} \times \chi^{2} A \times$
Wont: $B \tilde{y} = \lambda \tilde{y}$? $\tilde{y} = ?$
Attempt 1:
 $\tilde{y} = \chi \tilde{\chi}$
 $B \tilde{y} = \chi^{-1} A \times \chi \tilde{\chi}$

Eigenvalue Transformations (I)

What do the following transformations of the eigenvalue problem $A\mathbf{x} = \lambda \mathbf{x}$ do?

Shift. $A \rightarrow A - \sigma I = \mathbf{B}$

Inversion. $A \to A^{-1} - \mathcal{G}$ $A_{\mathcal{X}}^{-1} = \lambda \hat{\mathcal{X}} | A^{-1} \hookrightarrow \hat{\mathcal{X}}^{-1} = A^{-1} \hat{\mathcal{X}}$

 $B \stackrel{>}{\times} = A^{-1} \stackrel{>}{\times} = \frac{1}{x} \stackrel{>}{\times}$

Power. $A \rightarrow A^k$

 $A^{3}\ddot{x} = AA(\lambda A) - \lambda^{3}\ddot{x} \qquad A^{4}\ddot{x} = \lambda^{4}\ddot{x}$

Eigenvalue Transformations (II)

Polynomial
$$A \rightarrow aA^2 + bA + cI =$$

Similarity $T^{-1}AT$ with T invertible $A_{k=\lambda \hat{x}}^{\sim}$ $B \hat{y} = \lambda \hat{y}$. $\hat{y} = T^{-1} \hat{x}$

Sensitivity (I)

Assume A not defective. Suppose $X^{-1}AX = D$. Perturb $A \rightarrow A + E$. What happens to the eigenvalues?

Power Iteration

Demo: Motivating Power Iteration [cleared]

Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Further assume $\|\boldsymbol{x}_i\| = 1$.