- Dropping 1 exam, 1 hw, 2 quizzes
- Exam 2 ready to schedule (online + in person)
- HW 8: short
- Recitation section: Mondays @ 2:30 in Loomis 151 ↳ -3:30
  - ↳ BYOP
  - ↳ replaces 1 office hour
  - ↳ see class calendar
- HW poll

Goals:

- methods to compute one eigenvector
  ↳ "power method" / "power iteration"
  $A^{1,000} x_0$
- eigenvalues
- orthogonal iteration → QR iteration
- Schur form / Schur factorization

$A = X^{-1} D X$

$A^3 = X^{-1} D \cancel{X X^{-1}} D \cancel{X X^{-1}} D X = X^{-1} D^3 X$

$$y = A\left(A\left(A\left(A\left(A\,x\right)\right)\right)\right)$$

Power Iteration

Demo: Motivating Power Iteration [cleared]

Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.
Further assume $\|\boldsymbol{x}_i\| = 1$.

$A\alpha\vec{x} = \lambda\alpha\vec{x}$

$\vec{x}_0 =$ "something random"

$\vec{x}_i = A\vec{x}_{i-1} / \|A\vec{x}_{i-1}\| \quad (i = 1, 2, 3, \ldots)$

Assume $\vec{x}_0 = \alpha\vec{v}_1 + \beta\vec{v}_2 + \cdots$

$A\vec{v}_1 = \lambda_1\vec{v}_1$

$A\vec{v}_2 = \lambda_2\vec{v}_2$

$\vec{y} = A^{1000}\vec{x}_0 = A^{1000}(\alpha\vec{v}_1 + \beta\vec{v}_2) = \alpha\lambda_1^{1000}\vec{v}_1 + \beta\lambda_2^{1000}\vec{v}_2$

$$\frac{y}{\lambda_1^{1000}} = \alpha\vec{v}_1 + \beta\frac{\lambda_2^{1000}}{\lambda_1^{1000}}\vec{v}_2 = \alpha\vec{v}_1 + \beta\left(\frac{\lambda_2}{\lambda_1}\right)^{1000}\vec{v}_2$$

<1

$\vec{e}_i = \vec{x}_i - \vec{v}_1$

$$\vec{x}_0 = \sum_j \alpha_j \vec{v}_j$$

Assme $\|\vec{\alpha}\|_1 \leq 1$

Assme $\|\vec{v}_j\| = 1$

Moss, Sorry

$$\vec{e}_i = \vec{x}_i - \beta \vec{v}_1 = \frac{A x_{i-1}}{\lambda_1} = \frac{A^i}{\lambda_1^i} x_0 - \beta v_1$$

β: unknown scalar factor

$$= \sum_{j=1}^{n} \alpha_j \frac{\lambda_j^i}{\lambda_1^i} \vec{v}_j - \beta v_1$$

$$= \alpha_1 \vec{v}_1 - \beta \vec{v}_1 + \sum_{j=2}^{n} \alpha_j \Big(\underbrace{\frac{\lambda_j}{\lambda_1}}_{<1}\Big)^i \vec{v}_j$$

$$< \left(\frac{\lambda_2}{\lambda_1}\right)$$

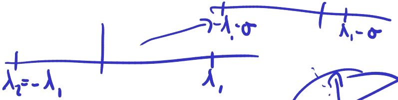
$$\leq \left(\frac{\lambda_2}{\lambda_1}\right)^i$$

error

$$\|e_i\| \leq \left(\frac{\lambda_2}{\lambda_1}\right)^i \|e_0\| \approx 1 \Leftarrow$$

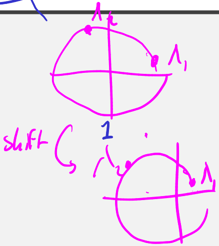
and $\beta = \alpha_1$

Power Iteration: Issues?



What could go wrong with Power Iteration?

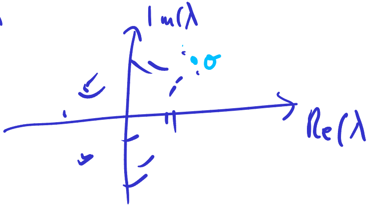
- What if $|\lambda_1| = |\lambda_2|$?
  - ↳ multiplicity
  - ↳ allow α to be complex! ρ
- What about complex eigenvalues?



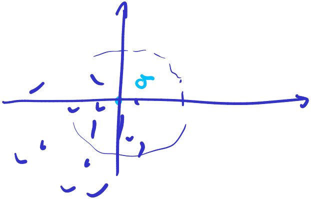
- $\vec{x}_0$ does not contain a component in $\vec{v}_1$

↳ rounding error will give you one, with time.

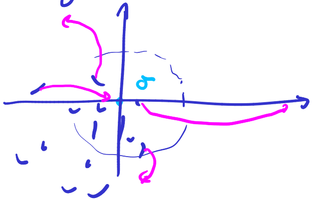
Eigenvalues of A



Eigenvalues of $A - \sigma I$



Eigenvalues of $(A - \sigma I)^{-1}$



What about Eigenvalues?

$\vec{x}$ $A\vec{x} = \lambda\vec{x}$

" $\frac{A\vec{x}}{\vec{x}}$ " "$\frac{\|Ax\|}{\|x\|}$"

Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

$$\frac{x^T A x}{x^T x} = \lambda$$

"Rayleigh quotient"

Rayleigh quotient iteration

Convergence of Power Iteration

What can you say about the convergence of the power method?

Say $\boldsymbol{v}_1^{(k)}$ is the kth estimate of the eigenvector $\boldsymbol{x}_1$, and

$$e_k = \left\| \boldsymbol{x}_1 - \boldsymbol{v}_1^{(k)} \right\|.$$