- AK office hours cancelled Thu
- Exam grades ? → Wed
- Exam 1 grades /18 not /20

QR Iteration/QR Algorithm

$$\bar{X}_0 = A$$
$$\bar{Q}_0 \bar{R}_0 = \bar{X}_0$$
$$\bar{X}_1 = \bar{R}_0 \bar{Q}_0$$

$$A^2 = \bar{Q}_0 \bar{Q}_1 \bar{R}_1 \bar{R}_0$$
$$\vdots$$
$$A^k = \dots$$

$\rightsquigarrow$ equivalent to orth. iter

$$\bar{X}_{k+1} = \bar{Q}_k^H \bar{Q}_{k-1}^H \cdots \bar{Q}_0^H A \underbrace{\bar{Q}_0 \cdots \bar{Q}_k}_{Q_{all}}$$

$Q \in \mathbb{R}^{n \times n}$

Q orthogonal $\Leftrightarrow Q^T Q = I$

$Q \in \mathbb{C}^{n \times n}$

Q unitary $\Leftrightarrow Q^H Q = I$

Proof sketch: Equivalence of QR iteration/Orth. iteration

Orthogonal Iteration (no bars)

- $X_0 := A$
  - $Q_0 R_0 := X_0$,
  - where we may choose $Q_0 = \bar{Q}_0$
  - $\hat{X}_0 = Q_0^H A Q_0 = Q_0^H Q_0 R_0 Q_0 = R_0 Q_0$
- $X_1 := A Q_0$
  - $Q_1 R_1 := X_1$, and because of $X_1 = Q_0 Q_0^H A Q_0 = Q_0 \bar{X}_1 = Q_0 \bar{Q}_1 \bar{R}_1$ we may choose $Q_1 = Q_0 \bar{Q}_1 = \bar{Q}_0 \bar{Q}_1$.
- $\vdots$

QR Iteration (with bars)

- $\bar{X}_0 := A$
  - $\bar{Q}_0 \bar{R}_0 := A$
- $\bar{X}_1 := \bar{R}_0 \bar{Q}_0 = \hat{X}_0$
  - $\bar{Q}_1 \bar{R}_1 := \bar{X}_1$
- $\bar{X}_2 := \bar{R}_1 \bar{Q}_1$
  - $\bar{X}_2 = Q_1^H A Q_1 = \hat{X}_1$
- $\vdots$

Demo: QR Iteration [cleared]

QR Iteration: Forward *and* Inverse

QR iteration may be viewed as performing inverse iteration. How?

$\bar{X}_0 = A$

$\bar{Q}_k \bar{R}_k = \bar{X}_k$

$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k$

$X_0^{-H} = A^{-H}$

lower triangular

$(\bar{Q}_k \bar{R}_k)^{-H} = \bar{Q}_k \bar{R}_k^{-H} = \bar{X}_k^{-H}$ QL factorization

$\bar{X}_{k+1}^{-H} = \bar{R}_k^{-H} \bar{Q}_k$

inverse conjugate $((\cdot)^{-1})^{H} = (-)^{-H}$

QR Iteration: Incorporating a Shift

How can we accelerate convergence of QR iteration using shifts?

$$\bar Q_k \bar R_k = \bar X_k - \sigma_k I \iff \bar R_k = \bar Q_k^H \bar X_k - \bar Q_k^H \sigma I$$

$$\bar X_{k+1} = \bar R_k \bar Q_k + \sigma_k I$$

$$\bar X_{k+1} = \bar R_k \bar Q_k + \sigma_k I = \left(\bar Q_k^H \bar X_k - \bar Q_k^H \sigma I\right)\bar Q_k + \sigma I$$

$$= \bar Q_k^H \bar X_k \bar Q_k^H - \underbrace{\cancel{Q_k^H Q_k \sigma}}_{I} + \cancel{I \sigma}$$

$$= Q_k^H \bar X_k Q_k$$

Demo: QR Iteration [cleared] (Shifted)

QR Iteration: Computational Expense

A full QR factorization at each iteration costs $O(n^3)$–can we make that cheaper?

$A = Q \left(\quad \right) Q^{-1}$

H

upper $\triangle$ +1

"upper Hessenberg"

start QR with H

↳ QR factorization in QR it: $O(n^2)$ $\quad \tilde{Q}_k \tilde{R}_k = X_k$

↳ ...? is this shape maintained? $\quad \tilde{X}_{k+1} = \tilde{R}_k \tilde{Q}_k$

Demo: Householder Similarity Transforms [cleared]

QR/Hessenberg: Overall procedure

Overall procedure:

1. Reduce matrix to Hessenberg form
2. Apply QR iteration using Givens QR to obtain Schur form

Why does QR iteration *stay* in Hessenberg form?

Assume $\bar{X}_k$ is Upper Hess

$\bar{Q}_k \bar{R}_k = \bar{X}_k$ $\qquad$ $\bar{Q}_k = \bar{X}_k R^{-1} =$ uH

$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k$

$\Rightarrow \bar{X}_{k+1}$ also uH.

What does this process look like for symmetric matrices?

Krylov space methods: Intro

What subspaces can we use to look for eigenvectors?

QR / orth. iter.: $\operatorname{span}(A^\ell \vec{y}_1, \dots, A^\ell \vec{y}_k)$

Krylov space

$$\operatorname{span}(\vec{x}, A\vec{x}, A^2\vec{x}, \dots, A^{k-1}\vec{x})$$

$$\bar{x}_0 \quad \bar{x}_1 \quad \bar{x}_{k+1}$$

$$K_k = \begin{pmatrix} | & & | \\ \vec{x}_0 & \cdots & x_{k-1} \\ | & & | \end{pmatrix} \leftarrow n \times k$$

Krylov for Matrix Factorization

What matrix factorization is obtained through Krylov space methods?

$$AK_n = \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix} = K_n \begin{pmatrix} 0 & & & 0 & \\ 1 & \ddots & & & \\ 0 & & \ddots & & K_n^{-1}x_n \\ \vdots & & \ddots & 0 & \\ 0 & & & 1 & \end{pmatrix}$$

$$K_n \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = x_1 \qquad = K_n \Big(\text{upper Hessenberg} \Big)$$

i

Assume invertibility → otherwise restart, messy

$$K_n^{-1} A K_n = H$$

just another sim. transform to upper Hess!

Conditioning in Krylov Space Methods/Arnoldi Iteration (I)

What is a problem with Krylov space methods? How can we fix it?

$$Q_n R_n = K_n \iff Q_n = K_n R_n^{-1}$$

$$Q_n^T A Q_n = R_n \underbrace{K_n^{-1} A K_n} R_n^{-1}$$

$$= H_n$$

Conditioning in Krylov Space Methods/Arnoldi Iteration (II)

$Q = (\; q_k \;)$

$$Q_n^T A Q_n = H \iff A Q_n = Q_n H$$

$$A\vec{q}_k = h_{1k}\vec{q}_1 + \dots + h_{k+1,k}\vec{q}_{k+1}$$

$$h_{jk} = \vec{q}_j^T A \vec{q}_k$$

Demo: Arnoldi Iteration [cleared] (Part 1)

Krylov: What about eigenvalues?

How can we use Arnoldi/Lanczos to compute eigenvalues?

Demo: Arnoldi Iteration [cleared] (Part 2)