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Lab problems
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Goal:
  * Krylov space for eigenvalue
    problems
  * SVD
  * nonlinear eqns.
Conditioning in Krylov Space Methods/Arnoldi Iteration (II)

\[ \text{span} \left\{ x_1, Ax_1, \ldots, A^{n-1}x_1 \right\} \]

\[ Q_n = \text{gram-schmidt} \left( x_1, Ax_1, \ldots, A^{n-1}x_1 \right) \]

\[ Q_n^T A Q_n = \begin{pmatrix} H \end{pmatrix}, \quad H = (h_{ij})_{ij} \]

\[ A \tilde{q}_k = h_1 \tilde{q}_1 + \cdots + h_{k+1} \tilde{q}_{k+1} \]

\[ h_{jk} = \tilde{q}_j A \tilde{q}_k \]

**Demo:** Arnoldi Iteration [cleared] (Part 1)
Krylov: What about eigenvalues?

How can we use Arnoldi/Lanczos to compute eigenvalues?

\[
Q_n = \begin{bmatrix}
\vdots \\
0 \\
\vdots 
\end{bmatrix} \quad Q_n^* A Q_n = \begin{bmatrix}
\vdots \\
0 \\
\vdots 
\end{bmatrix}
\]

\[
A - A^+ = (Q_n^* A Q_n)^r
\]

"Ritz values"

A symmetric \( \Rightarrow \) tridiagonal "Lanczos"
Computing the SVD (Kiddy Version)

\[ A = U \Sigma V^T \]

\[
A^T A \mathbf{V} = \mathbf{V} \mathbf{D} \quad \text{diag.}
\]

\[
A^T A = U \Sigma^2 U^T
\]

\[
A = U \Sigma V^T \Rightarrow \text{solve } U \Sigma = AV
\]

**Demo:** Computing the SVD [cleared]

“Actual”/“non-kiddy” computation of the SVD:

- Bidiagonalize \( A = U \begin{bmatrix} B & \mathbf{0} \end{bmatrix} V^T \), then diagonalize via variant of QR.

- References: Chan ’82 or Golub/van Loan Sec 8.6.
Outline

Introduction to Scientific Computing

Systems of Linear Equations

Linear Least Squares

Eigenvalue Problems

Nonlinear Equations
  Introduction
  Iterative Procedures
  Methods in One Dimension
  Methods in $n$ Dimensions ("Systems of Equations")

Optimization

Interpolation

Numerical Integration and Differentiation

Initial Value Problems for ODEs

Boundary Value Problems for ODEs

Partial Differential Equations and Sparse Linear Algebra

Fast Fourier Transform

Additional Topics
What is the goal here?

\[ f(x) = 0 \quad f: \mathbb{R}^n \to \mathbb{R}^n \]
Showing Existence

How can we show existence of a root?

1. Intermediate value thm.

2. Inverse function theorem
   - If $f$ is invertible at $x_0$, in a neighborhood of $x_0$, $f(x) = 0$. Contracation mapping thm.
If: \( g: \mathbb{R}^n \to \mathbb{R}^n \)

\[ \forall x, y \in S : \| g(x) - g(y) \| < \gamma \| x - y \| \quad 0 \leq \gamma < 1 \]

Then: There exists a fixed point \( g(k) = k \)

(5) fixed point iteration
Sensitivity and Multiplicity

What is the sensitivity/conditioning of root finding?

\[
\text{cond (root finding)} = \text{cond (evaluating inverse } p^{-1}(0))
\]

\[p(x) = 0 \Rightarrow \text{need to use absolute cond. or }\]

What are multiple roots?

\[p(x) = 0 \quad p'(x) \neq 0 \quad p^{(n-1)}(x^*) = 0\]

How do multiple roots interact with conditioning?

inverse function is steep near one; body conditioned.
Rates of Convergence

What is linear convergence? quadratic convergence?

An iterative converges with rate $r$ if

$$\lim_{k \to \infty} \frac{\|e_k\|}{\|e_{k-1}\|} = C \{ r \} < \infty$$

Limit: ignore finite step behavior.

Power it: linear

Rayleigh QI: quadratically
About Convergence Rates

**Demo:** Rates of Convergence [cleared]
Characterize linear, quadratic convergence in terms of the ‘number of accurate digits’.

- *linear*: a reliable
  - a kind of slow
  - o ind. of starting point

- *superlinear* ($r > 1$):  
  - only converges once $\|e_k\|$ is small enough
  - fast.
Stopping Criteria

Comment on the ‘foolproof-ness’ of these stopping criteria:

1. $|f(x)| < \varepsilon$ (‘residual is small’)
2. $\|x_{k+1} - x_k\| < \varepsilon$
3. $\|x_{k+1} - x_k\| / \|x_k\| < \varepsilon$
Bisection Method

**Demo:** Bisection Method [cleared]

What's the rate of convergence? What's the constant?

Linear, with constant $\frac{1}{2}$