Exam Jupyterlab:
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No OH today

Exam 2 grades

Time refund!

Jlab problems
→ shift + reload

Goal:

- Krylov space for eigenvalue / lin sys?
- SVD
- nonlinear eq'ns.

# Conditioning in Krylov Space Methods/Arnoldi Iteration (II)

$$\operatorname{span}\{\vec{x}, A\vec{x}, \dots, A^{n-1}\vec{x}\}$$

$$Q_n = \text{gram-schmidt}(x, Ax, \dots, A^{n-1}x)$$

$$Q_n^T A Q_n = \quad H = (h_{ij})_{ij}$$

$$A\vec{q}_k = h_{1k}\vec{q}_1 + \dots + h_{k+1,k}\,\vec{q}_{k+1}$$

$$h_{jk} = \vec{q}_j A \vec{q}_k$$

**Demo:** Arnoldi Iteration [cleared] (Part 1)

## Krylov: What about eigenvalues?

How can we use Arnoldi/Lanczos to compute eigenvalues?

$Q_n = \begin{bmatrix} & \\ & \end{bmatrix}$

$Q_n^T A Q$

$A = A^T$

$(Q_n^T A Q_n)^T$

$H = Q_n^T A Q_n = \begin{bmatrix} & \\ & \end{bmatrix}$

"Ritz values"

A symmetric: $H$ tridiagonal "Lanczos"

**Demo:** Arnoldi Iteration [cleared] (Part 2)

# Computing the SVD (Kiddy Version)

$A = U\Sigma V^T$

$\underbrace{A^TA}V = VD$ diag.

baaad!

$A = U\Sigma V^T \Rightarrow$ solve $U\Sigma = AV$

$A^TA = V\Sigma \cancel{U^TU} \Sigma V^T$

$= V\Sigma^2 V^T$

$D$

**Demo:** Computing the SVD [cleared]

"Actual"/"non-kiddy" computation of the SVD:

- Bidiagonalize $A = U \begin{bmatrix} B \\ 0 \end{bmatrix} V^T$, then diagonalize via variant of QR.
- References: Chan '82 or Golub/van Loan Sec 8.6.

# Outline

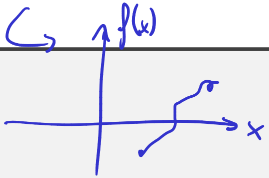
# Solving Nonlinear Equations

What is the goal here?

$$f(\vec{x}) = \vec{0} \qquad f: \mathbb{R}^n \to \mathbb{R}^n$$

# Showing Existence

$f: \mathbb{R}^1 \to \mathbb{R}^1$




How can we show existence of a root?

1D: Intermediate value thm.

nD: Inverse function theorem $\quad f(\vec{x}+\vec{h}) = f(\vec{x}) + J_f(\vec{x})\,\vec{h} + \dots$

If $J_f$ invertible at $\vec{x}$, in a neighborhood of $\vec{x}$, $f(\vec{x}) = 0$.

Contraction mapping thm.

If: $g: \mathbb{R}^n \to \mathbb{R}^n$

$\forall \vec{x}, \vec{y} \in S: \| g(\vec{x}) - g(\vec{y}) \| \leq \gamma \| \vec{x} - \vec{y} \| \quad 0 \leq \gamma < 1$

closed bdd, $g(S) \in S$

Then: There exists a fixed point $g(\vec{x}^*) = \vec{x}^*$

↳ fixed point iteration

# Sensitivity and Multiplicity

What is the sensitivity/conditioning of root finding?

cond (root finding) = cond (evaluating inverse $f^{-1}(0)$)

$f(x^*)=0 \Rightarrow$ need to use absolute cond. nr.

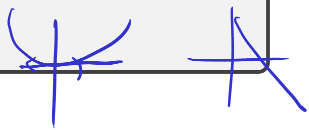
What are multiple roots? of multiplicity $k$

$f(x^*)=0$

$\vdots$

$f^{(k-1)}(x^*)=0$

$f(x) = (x-x_1)^2 \dots \quad \dots$



How do multiple roots interact with conditioning?

inverse function is steep near one; badly conditioned.

# Rates of Convergence

power it $\|e_k\| \leq C\,\|e_{k-1}\|$

$0 \leq C < 1$

What is *linear convergence*? *quadratic convergence*?

An iterative converges with rate $r$ :⟺ $\|e_k\| \leq C\,\|e_{k-1}\|^r$

$$\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C \begin{cases} >0 \\ <\infty \end{cases}$$

limit: ignore finite step behavior

Power it : linear

Rayleigh QI : quadratically

# About Convergence Rates

**Demo:** Rates of Convergence [cleared]

Characterize linear, quadratic convergence in terms of the 'number of accurate digits'.

linear:
- reliable
- kind of slow
- ind. of starting point

superlinear ($r>1$):
- only converges once $\|e_k\|$ is "small enough"
- fast.

## Stopping Criteria

Comment on the 'foolproof-ness' of these stopping criteria:

1. $|f(x)| < \varepsilon$ ('residual is small')
2. $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| < \varepsilon$
3. $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| / \|\boldsymbol{x}_k\| < \varepsilon$

# Bisection Method

**Demo:** Bisection Method [cleared]

What's the rate of convergence? What's the constant?

linear, with constant 1/2