Review

- Contractive map
  \[ |g(x) - g(y)| / |x - y| \leq C |x - y| \]
  \[ 0 < C < 1 \]

- Rates of conv.
  - Linear conv.
    \[ e_k \to 0 \quad e_k = x_k - x^* \]

  \[ \|e_k\| \leq C \|e_{k+1}\| \]

  - Quadratic conv.
    \[ \|e_k\| \leq C \|e_{k+1}\|^2 \]

  True def. has a limit.

Goals

- FPI
- Newton (1D)
- Secant method
- Advection method
- Newton (nD)

Example: Bisection
Fixed Point Iteration

\[ x_0 = \text{starting guess} \]
\[ x_{k+1} = g(x_k) \]

**Demo:** Fixed point iteration [cleared]

When does fixed point iteration converge? Assume \( g \) is smooth.

**Claim:** Converges (at least locally) if \( |g'(x^*)| < 1 \)

Let \( x^* \) be the fixed point, i.e., \( g(x^*) = x^* \).

\[ e_{n+1} = x_{n+1} - x^* \]
\[ e_{n+1} = x_n - x^* \quad \text{gives} \quad g(x_n) - g(x^*) \]
Fixed Point Iteration: Convergence cont’d.

Error in FPI: $e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$

$e_{k+1} = g(x_k) - g(x^*) = g'(\theta_k)(x_k - x^*) = g'(\theta_k)e_k$

$g(x_k) = g(x^*) + g'(x^*) \cdot (x_k - x^*) + \frac{g''(\gamma)}{2} (x_k - x^*)^2 + \cdots$

$g(x_k) = g(x^*) + g'(\theta_k) \cdot (x_k - x^*) + \frac{g''(\theta_k)}{2} (x_k - x^*)^2$

$g(x_k) = g(x^*) + g'(\theta_k) \cdot (x_k - x^*)$ if $\theta_k \in [x^*, x_k]$

What if $g'(x^*) = 0$?

$e_{k+1} = g''(\gamma_k)(x_k - x^*)^2 \leq g''(\gamma_k)e_k^2$

$e_{k+1} = g''(\gamma_k)(\theta_k - x_k)(x_k - x^*)^2 \leq g''(\gamma_k)e_k^2$
Newton’s Method

Derive Newton’s method.

\[ p(x^*) = 0 \]

\[ p(x_{k+1}) \approx p(x_k) + p'(x_k) h = \frac{p(x_k)}{p'(x_k)} \]

\[ 0 = \frac{p(x_k)}{p'(x_k)} - p(x_k) + p'(x_k) h \]

\[ \Rightarrow -\frac{p(x_k)}{p'(x_k)} = h \]

\[ x_{k+1} = x_k - \frac{p(x_k)}{p'(x_k)} = g(x_k) \]

**Demo:** Newton’s method [cleared]

If \( p(x^*) = 0 \), then \( q(x^*) = x^* - \frac{0}{p'(x^*)} = x^* \)

\( x^* \) is a fixed pt. of Newton f. for some b.v.!
Convergence and Properties of Newton

What’s the rate of convergence of Newton’s method?

\[ g'(x) = \frac{f(x) f''(x)}{f'(x)^2} \]

Criterion for quadratic convergence: \( g'(x^*) = 0 \) if \( f(x^*) = 0 \) implies single root.

Drawbacks of Newton?

Demo: Convergence of Newton’s Method [cleared]
Convergence and Properties of Newton

What’s the rate of convergence of Newton’s method?

**Drawbacks** of Newton?

- Convergence argument only good *locally*
  Will see: convergence only local (near root)
- Have to have derivative!

**Demo:** Convergence of Newton’s Method [cleared]
Secant Method

What would Newton without the use of the derivative look like?

\[ f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = s_n \]

\[ x_{n+1} = x_n - \frac{f(x_n)}{s_n} \]
Convergence of Properties of Secant

Rate of convergence is \((1 + \sqrt{5}) / 2 \approx 1.618\). (proof)

Drawbacks of Secant?

Demo: Secant Method [cleared]
Demo: Convergence of the Secant Method [cleared]

Secant (and similar methods) are called Quasi-Newton Methods.
Convergence of Properties of Secant

Rate of convergence is \((1 + \sqrt{5})/2 \approx 1.618\). (proof)

Drawbacks of Secant?

- Convergence argument only good \textit{locally}
  Will see: convergence only local (near root)
- Slower convergence
- Need two starting guesses

Demo: Secant Method [cleared]
Demo: Convergence of the Secant Method [cleared]

Secant (and similar methods) are called \textit{Quasi-Newton Methods}. 
Improving on Newton?

How would we do “Newton + 1” (i.e. even faster, even better)?