Review

- contractive map

$$|g(x) - g(y)| \leq C|x-y|$$

$$0 \leq C < 1$$

- rates of conv.

linear conv.

$e_k \to 0 \qquad e_k = x_k - x^*$

$\|e_{k+1}\| \leq C \cdot \|e_k\|$ ⟶ example: bisection

quadratic conv.

$$\|e_{k+1}\| \leq C\,\|e_k\|^2$$

true def. has a lim.

Goals

- FPI
- Newton (1D)
- secant method
- adventurous methods
- Newton (nD)

Fixed Point Iteration

$$\begin{aligned} x_0 &= \langle\text{starting guess}\rangle \\ x_{k+1} &= g(x_k) \end{aligned}$$

Demo: Fixed point iteration [cleared]

When does fixed point iteration converge? Assume g is smooth.

Claim: converges (at least locally) if $|g'(x^*)| < 1$

Let x^* be the fixed point, i.e. $g(x^*) = x^*$.

$e_k = x_k - x^*$

$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$

Fixed Point Iteration: Convergence cont'd.

Error in FPI: $e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$

$e_{k+1} = g(x_k) - g(x^*) = g'(\theta_k)(x_k - x^*) = g'(\theta_k) e_k$

$g(x_k) = g(x^*) + g'(x^*)\cdot(x_k - x^*) + \frac{g''(x^*)}{2}(x_k - x^*)^2 + \dots$

$g(x_k) = g(x^*) + g'(x^*)\cdot(x_k - x^*) + \frac{g''(\theta_k)}{2}(x_k - x^*)^2$

$g(x_k) = g(x^*) + g'(\theta_k)\cdot(x_k - x^*)$ $\theta_k \in [x^*, x_k]$

What if $g'(x^*) = 0$?

$g'(x) \approx \cancel{g'(x^*)} + g''(\eta_k)\underbrace{(x - x^*)}_{e_k}$

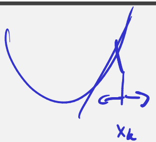
$e_{k+1} = g''(\eta_k)(\theta_k - x^*)(x_k - x^*) \leq g''(\eta_k) e_k^2$

$\leq |x_k - x^*|$

Newton's Method

$f(x^*) = 0$

Derive Newton's method.



$$f(x_k + h) \approx f(x_k) + f'(x_k)h = \tilde{f}_k(h)$$

$$0 = \tilde{f}_k(h) = f(x_k) + f'(x_k)h$$

$$\Leftrightarrow \; -\frac{f(x_k)}{f'(x_k)} = h$$

$$x_{k+1} = x_k + h = x_k - \frac{f(x_k)}{f'(x_k)} = g(x_k)$$

Demo: Newton's method [cleared]

If $f(x^*) = 0$,

then $g(x^*) = x^* - \frac{0}{f'(x^*)} = x^*$

$\Rightarrow x^*$ is a fixed pt. of Newton for single roots!

Convergence and Properties of Newton

What's the rate of convergence of Newton's method?

$$g'(x) = \frac{f(x)\,f''(x)}{f'(x)^2} \qquad \left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

Criterion for quadr. conv: $g'(x^*) = 0$.

$f(x^*)=0 \Rightarrow g'(x^*)=0$ if single root.

Drawbacks of Newton?

Demo: Convergence of Newton's Method [cleared]

Convergence and Properties of Newton

What's the rate of convergence of Newton's method?

Drawbacks of Newton?

- Convergence argument only good *locally*
  Will see: convergence only local (near root)
- Have to have derivative!

Demo: Convergence of Newton's Method [cleared]

Secant Method

What would Newton without the use of the derivative look like?

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = s_k$$

$$x_{k+1} = x_k - \frac{f(x_k)}{s_k}$$

Convergence of Properties of Secant

Rate of convergence is $\left(1+\sqrt{5}\right)/2 \approx 1.618$. (proof)

Drawbacks of Secant?

Demo: Secant Method [cleared]
Demo: Convergence of the Secant Method [cleared]

Secant (and similar methods) are called Quasi-Newton Methods.

Convergence of Properties of Secant

Rate of convergence is $\left(1+\sqrt{5}\right)/2 \approx 1.618$. (proof)

Drawbacks of Secant?

- Convergence argument only good *locally*
  Will see: convergence only local (near root)
- Slower convergence
- Need two starting guesses

Demo: Secant Method [cleared]
Demo: Convergence of the Secant Method [cleared]

Secant (and similar methods) are called Quasi-Newton Methods.

Improving on Newton?

How would we do "Newton + 1" (i.e. even faster, even better)?