Announcements

ICES

4CH1 bumped to Dec 6

Final content cutoff today

Recitation session (Monday 2:30)

Exam 3 page grades: likely this weekend

Review

Newton - Cotes

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\, w_i$$

method of undet. coeff.

$$V^T \vec{w} = \left(\int_a^b x^i\,dx\right)_i$$

nodes go here

Cheby nodes + basis: Clenshaw-Curtis

todd node cont

Goals

- Gauss
- Composite quadrature (mainly accuracy)
- derivatives
- methods for that

Gaussian Quadrature

So far: nodes chosen from outside.
Can we gain something if we let the quadrature rule choose the nodes, too? Hope: More design freedom → Exact to higher degree.

Could: set up method of undet. coeff. with unknown nodes → yucky, nonlinear

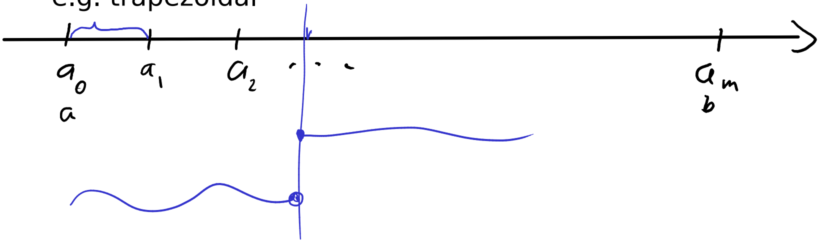
Demo: Gaussian quadrature weight finder [cleared]

Composite Quadrature

High-order polynomial interpolation requires a high degree of smoothness of the function.

Idea: Stitch together multiple lower-order quadrature rules to alleviate smoothness requirement.

e.g. trapezoidal



Error in Composite Quadrature

What can we say about the error in the case of composite quadrature?

$$\left|\int f - p_{n-1}\right| \le C \cdot \|f^{(n)}\|_\infty h^{n+1} \quad (h \to 0)$$

$$\left|\int_a^b f(x)\,dx - \sum_{j=1}^{m}\sum_{i=1}^{n} w_{i,j} f(x_{i,j})\right| \le C\,\|f^{(n)}\|_\infty \sum_{j=1}^{m} (\overbrace{a_j - a_{j-1}}^{h})^{n+1}$$

$$= C\,\|f^{(n)}\|_\infty \sum_{j=1}^{m} \underbrace{(a_j - a_{j-1})^n}_{h^n} (a_j - a_{j-1}) = C\,\|f^{(n)}\|_\infty h^n \sum_{j=1}^{m} (a_j - a_{j-1})$$

$$= C\,\|f^{(n)}\|_\infty h^n (a_m - a_0)$$

$m \sim \frac{1}{h}$

$a_0 \quad h \quad a_1 \quad a_2 \quad a_3 \quad a_m$

Composite Quadrature: Notes

Observation: Composite quadrature loses an order compared to non-composite.

Idea: If we can estimate errors on each subinterval, we can shrink (e.g. by splitting in half) only those contributing the most to the error. (adaptivity)

Taking Derivatives Numerically

Why *shouldn't* you take derivatives numerically?

$f(x) = e^{i\alpha x}$ $\max|f(x)| \le 1$ $\max|f'| = \alpha$

conditioning of derivatives is arbitrarily bad

catastrophic cancellation $\longrightarrow$ $\dfrac{\underbrace{f(x+h) - f(x)}}{h}$ $(h \to 0)$

$= \frac{1}{h} f(x+h) - \frac{1}{h} f(x)$

Numerical Differentiation: How?

How can we take derivatives numerically?

$\vec{x} = (x_i)_{i=1}^{n}$

$V_{ij} = \varphi_j(x_i)$

$\vec{\alpha} = V^{-1} f(\vec{x})$

$p_{n-1}(x) = \sum_{i=1}^{n} \alpha_i \varphi_i(x)$

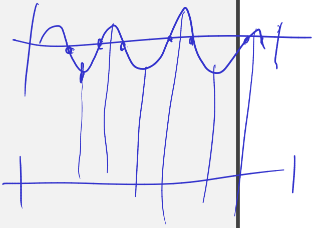
$p_{n-1}'(x) = \sum_{i=1}^{n} \alpha_i \varphi_i'(x)$

Demo: Taking Derivatives with Vandermonde Matrices [cleared] (Basics)

Numerical Differentiation: Accuracy

How accurate is numerical differentiation (with a polynomial basis)?

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi(x))}{n!} \prod_{i=1}^{n} (x - x_i)$$



$$f'(x) - p'_{n-1}(x) \approx \frac{f^{(n)}(\xi)}{n!} \underbrace{\left(\prod_{i=1}^{n} (x - x_i) \right)'}_{\underbrace{\prod_{i=1}^{n-1} (x - e_i)}_{\le h^{n-1}}}$$

Demo: Taking Derivatives with Vandermonde Matrices [cleared]

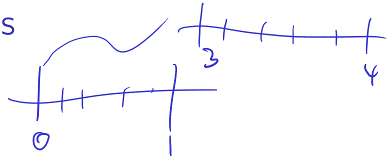
Differentiation Matrices

How can numerical differentiation be cast as a matrix-vector operation?

$$D = V'V^{-1}$$

Demo: Taking Derivatives with Vandermonde Matrices [cleared] (Build D)

Properties of Differentiation Matrices

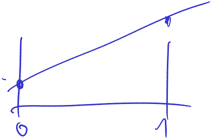


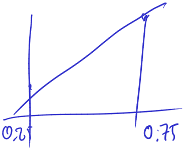
How do I find second derivatives?

D^2

Does D have a nullspace?

constants





Numerical Differentiation: Shift and Scale

Does D change if we shift the nodes $(x_i)_{i=1}^n \to (x_i + c)_{i=1}^n$?

it does not

Does D change if we scale the nodes $(x_i)_{i=1}^n \to (\alpha x_i)_{i=1}^n$?

$D_{\alpha \vec{x}} = D_{\vec{x}} / \alpha$

Finite Difference Formulas from Diff. Matrices

How do the rows of a differentiation matrix relate to FD formulas?

$f(\vec{x})$

$$Df(\vec{x}) \approx \begin{pmatrix} f'(x_1) \\ \vdots \\ f'(x_n) \end{pmatrix}$$

Assume a large equispaced grid and 3 nodes w/same spacing. How to use?

Finite Differences: via Taylor