fully error scenario

input/output error scenario

input error

\[ x \xrightarrow{P} \hat{y}(x) \]

\[ \hat{y}(x) \xrightarrow{f} y(x) \]

\[ y(x) \xrightarrow{f} f(x) \]

\[ f(x) \xrightarrow{f} f(k) \]

\[ f(k) \xrightarrow{e\text{nd\ error}} \]
Sensitivity and Conditioning

Consider a more general setting: An input $x$ and its perturbation $\hat{x}$.

\[
\frac{|f(x) - f(x')|}{|f(x)|} \leq \kappa_{rel} \cdot \frac{|x - x'|}{|x|}
\]

\[
\max \frac{|f(x) - f(x')|}{|f(x)|} \leq \kappa_{rel} \cdot \frac{|x - x'|}{|x|}
\]

\{ \text{For which } x, \hat{x}? \text{ (depends! All scenarios can be relevant)} \}

\{(x, \hat{x}) \in S\}
Can you also define an *absolute* condition number?
Can you also define an *absolute* condition number?

Certainly:

\[ \kappa_{\text{abs}} = \max_{x, \hat{x}} \frac{|f(x) - f(\hat{x})|}{|x - \hat{x}|} \]

But: less commonly used than relative, because we *typically* care about relative error.

When not specified: Assume condition number means *relative*. 
Interpreting a Condition Number

What does it mean for condition numbers to be small/large?

\[
\text{cond nr. } 5 \leq \text{cond nr. } 5000
\]

Relate the (relative) condition number back to the setting of (relative) backward error.
Interpreting a Condition Number

What does it mean for condition numbers to be small/large?

If the condition number is...
▶ small: the problem *well-conditioned* or insensitive
▶ large: the problem *ill-conditioned* or sensitive

*Can* also talk about condition number for a single input $x$.

Relate the (relative) condition number back to the setting of (relative) backward error.

\[ \text{rel. output error} \leq \kappa \cdot \text{rel. input error} \]
Example: Condition Number of Evaluating a Function

\[ y = f(x). \] Assume \( f \) differentiable.

\[ r_c = \max_{\Delta x} \frac{|\Delta y|}{|\Delta x|} \cdot \frac{|y|}{|\Delta y|} \]

\[ \Delta y = f(x + \Delta x) - f(x) = f'(x) \Delta x \approx f'(x) \Delta x \]

\[ K = \frac{|\Delta y|}{|\Delta x|} \cdot \frac{|y|}{|\Delta y|} \approx \frac{|f'(x)|}{|f(x)|} \cdot \frac{|\Delta x|}{|\Delta y|} = \frac{|x \cdot f'(x)|}{|f(x)|} \]

**Demo:** Conditioning of Evaluating \( \tan \) [cleared]
Stability and Accuracy

Previously: Considered *problems or questions*.
Next: Considered *methods*, i.e. computational approaches to find solutions.
When is a method *accurate*?

When is a method *stable*?
Stability and Accuracy

Previously: Considered *problems* or *questions.*
Next: Considered *methods*, i.e. computational approaches to find solutions.
When is a method *accurate*?

Closeness of method output to true answer for unperturbed input.

When is a method *stable*?

- “A method is stable if the result it produces is the exact answer for a nearby input.”
- The above is commonly called *backward stability* and is a stricter requirement than just the temptingly simple:

If the method’s sensitivity to variation in the input is no (or not much) greater than that of the problem itself.
How can I produce inaccurate results?
Getting into Trouble with Accuracy and Stability

How can I produce inaccurate results?

- Apply an inaccurate method
- Apply an unstable method to a well-conditioned problem
- Apply any type of method to an ill-conditioned problem
In-class activity: Forward/Backward Error

\[
x \cdot \frac{f'(x)}{f(x)} = \frac{x \cdot \cos x}{\sin x}
\]

\(C_0, \frac{\pi}{2}\)
Wanted: Real Numbers... in a computer

Computers can represent *integers*, using bits:

\[ 23 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = (10111)_2 \]

How would we represent fractions?

\[ 0.625 = \left( 1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3} \right) \]

- **Fixed-point number expr.**
- very small
- very big

**Downsides:**
another downside of fixed point
amount of roundoff error
depends on magnitude

For ex., a number that occupies 27 bits

\[
\frac{22}{6}
\]

approx roundoff error: \(2^{-17} = \frac{|real| \text{ error}}{|number|}\)

23. 6.25 = \[
\frac{24}{1} = 2^4
\]
Convert $13 = (1101)_2$ into floating point representation.

What pieces do you need to store an FP number?