Exam 1 in about two weeks:

Go to prairietest.org to schedule

HW 2 due last night

HW 3

Please use the helpdesk feature!

Matrix norms

$\|\vec{x}\|_{1,2,\infty}$

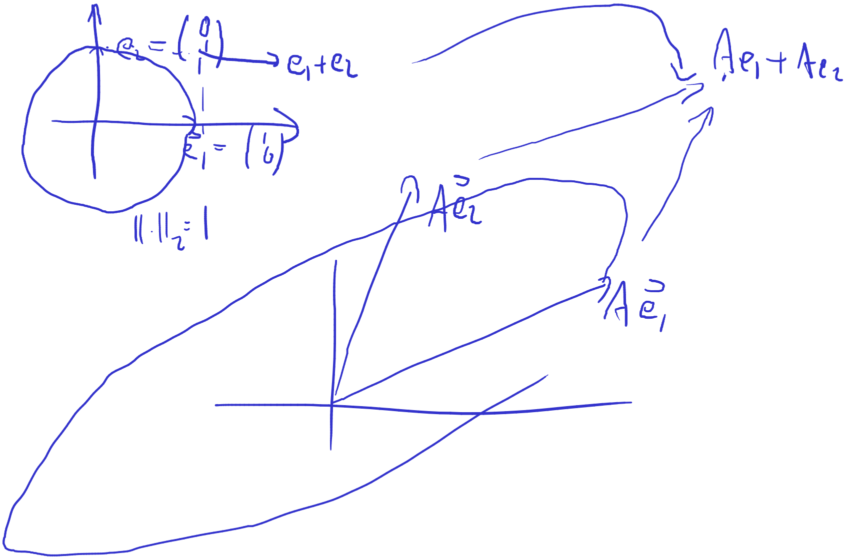
"submultiplicativity"

$\|A\vec{x}\|_v \le \|A\|_M \|\vec{x}\|_v$

$$\underbrace{\max_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_v}{\|\vec{x}\|_v}} =: \|A\|_m$$

mat. norm induced by vector norm $\|\cdot\|_v$.

<u>Goal</u>: - build on m.n. to get
- conditioning bounds
- error bounds for changed matrix



$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$e_1 + e_2$
$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\|\cdot\|_2 = 1$
$Ae_1 + Ae_2$
$A\vec{e}_2$
$A\vec{e}_1$

Identifying Matrix Norms

What is $\|A\|_1$? $\|A\|_\infty$?

$$\|A\|_1 = \max_j \sum_i |A_{ij}| \qquad \|A\|_\infty = \max_i \sum_j |A_{ij}|$$

How do matrix and vector norms relate for $n \times 1$ matrices?

Demo: Matrix norms [cleared]

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

- $\|A\| > 0 \Leftrightarrow A \neq 0$.
- $\|\gamma A\| = |\gamma| \, \|A\|$ for all scalars γ.
- Obeys triangle inequality $\|A + B\| \leq \|A\| + \|B\|$

But also some more properties that stem from our definition:

$$\|Ax\| \leq \|A\| \, \|x\|$$

$$\|AB\| \leq \|A\| \, \|B\|$$

Conditioning

What is the condition number of solving a linear system $A\boldsymbol{x} = \boldsymbol{b}$?

out in

$$A(\vec{x}+\Delta\vec{x}) = \vec{b}+\Delta\vec{b}. \Rightarrow A\Delta\vec{x} = \Delta\vec{b}$$

$\Delta\vec{x}$ want

$\Delta\vec{b}$ have

$$\frac{\text{rel. err in output}}{\text{rel. err. in input}} = \frac{\|\Delta x\| / \|x\|}{\|\Delta b\| / \|b\|} = \frac{\|\Delta x\| \; \|b\|}{\|\Delta b\| \; \|x\|}$$

$$= \frac{\|A^{-1}\Delta b\| \; \|Ax\|}{\|\Delta b\| \; \|x\|} \le \|A^{-1}\| \, \|A\| \frac{\|\Delta b\| \; \|x\|}{\|\Delta b\| \; \|x\|}$$

matrix cond. nr. $\kappa(A)$

showed: $\kappa(A)$ is an upper bound for cond. of lin. system solving

To show sharpness: need examples $\to$ hw3

Actually: <u>is</u> the cond number

Conditioning of Linear Systems: Observations

Showed $\kappa(\text{Solve } A\boldsymbol{x} = \boldsymbol{b}) \leq \|A^{-1}\| \|A\|$.
I.e. found an *upper bound* on the condition number. With a little bit of fiddling, it's not too hard to find examples that achieve this bound, i.e. that it is *sharp*.

So we've found the *condition number of linear system solving*, also called the condition number of the matrix A:

$$\operatorname{cond}(A) = \kappa(A) = \|A\| \left\|A^{-1}\right\|.$$

Conditioning of Linear Systems: More properties

$\kappa(A) = \|A\| \, \|A^{-1}\|$

- cond is relative to a given norm. So, to be precise, use

$$\underset{\kappa}{\mathrm{cond}_2} \quad \text{or} \quad \underset{\kappa}{\mathrm{cond}_\infty}.$$

$\kappa(A) = \|A\| \, \|A^{-1}\|$

- If A^{-1} does not exist: $\mathrm{cond}(A) = \infty$ by convention.

Assume invertibility

What is $\kappa(A^{-1})$?

$\kappa(A)$

solve

$A\vec{x} = \vec{b}$

What is the condition number of matrix-vector multiplication?

$\kappa(A)$

matvec:

$A\vec{x} = \vec{b}$

$\Leftrightarrow \; A^{-1}\vec{b} = \vec{x}$

Demo: Condition number visualized [cleared]

Demo: Conditioning of 2x2 Matrices [cleared] matrix: $\quad A\vec{x} = \vec{b}$

in $\quad$ out

$$\frac{\|\Delta b\|}{\|b\|} \leq \kappa(A) \frac{\|\Delta x\|}{\|x\|}$$

Residual Vector

What is the residual vector of solving the linear system

$$\boldsymbol{b} = A\boldsymbol{x}?$$

$\hat{x}$ some proposed sol.

$$\vec{b} - A\hat{\vec{x}} = \vec{r}$$

residual vector is computable

How close is x to $\hat{x}$?

Residual and Error: Relationship

How do the (norms of the) residual vector $\boldsymbol{r}$ and the error $\Delta \boldsymbol{x} = \boldsymbol{x} - \widehat{\boldsymbol{x}}$ relate to one another?

$$\|\Delta\vec{x}\| = \|\vec{x} - \hat{\vec{x}}\| = \|\underbrace{A^{-1}A}_{\mathrm{id}}(\vec{x} - \hat{\vec{x}})\| = \|A^{-1}(\vec{b} - A\hat{\vec{x}})\|$$

$$= \|A^{-1}\vec{r}\|$$

$$\frac{\|\Delta\vec{x}\|}{\|\hat{\vec{x}}\|} = \frac{\|A^{-1}\vec{r}\|}{\|\hat{\vec{x}}\|} \le \frac{\|A^{-1}\|\,\|\vec{r}\|}{\|\hat{\vec{x}}\|} \le \kappa(A)\frac{\|\vec{r}\|}{\|A\|\,\|\hat{\vec{x}}\|}$$

$$\le \kappa(A)\frac{\|\vec{r}\|}{\|A\hat{\vec{x}}\|}$$

Changing the Matrix

So far, only discussed changing the RHS, i.e. $A\boldsymbol{x} = \boldsymbol{b} \quad \rightarrow \quad A\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{b}}$.
The matrix consists of FP numbers, too—it, too, is approximate. I.e.

$$A\boldsymbol{x} = \boldsymbol{b} \quad \rightarrow \quad \widehat{A}\widehat{\boldsymbol{x}} = \boldsymbol{b}.$$

What can we say about the error due to an approximate matrix?