October 3, 2024

Announcements

- Exam 2

Goals

- QR
  - ↳ G-S
  - ↳ Householder
  - ↳ Givens
- Rank-deficient LSQ
  - ↳ SVD

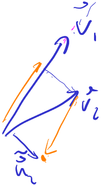
Review

$\vec q_1, \vec q_2, \dots, \vec q_n \in \mathbb{R}^n$ orthonormal

$\vec q_i \cdot \vec q_j = 0 \quad (i \neq j) \qquad \vec q_i \cdot \vec q_i = 1 = \|q_i\|_2^2$

$\vec v = \underbrace{\vec q_1 \cdot \vec v}\, \vec q_1 + \underbrace{\quad}\, \vec q_2 + \cdots + \underbrace{\quad}\, \vec q_n$

$Q^T = Q^{-1}$



$\vec u_1 = \vec v_1 / \|\vec v_1\|_2$

$\tilde{\vec u}_2 = \vec v_2 - (\vec u_1 \cdot \vec v_2)\, u_1$

$\vec u_2 = \dfrac{\tilde u_2}{\|\tilde u_2\|}$

Computing QR

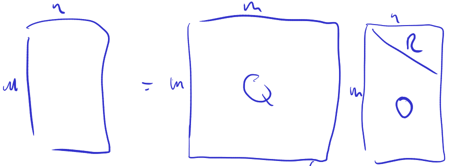
- Gram-Schmidt
- Householder Reflectors
- Givens Rotations

Demo: Gram-Schmidt–The Movie [cleared] (shows *modified G-S*)
Demo: Gram-Schmidt and Modified Gram-Schmidt [cleared]
Demo: Keeping track of coefficients in Gram-Schmidt [cleared]
Seen: Even modified Gram-Schmidt still unsatisfactory in finite precision arithmetic because of roundoff.

NOTE: Textbook makes further modification to 'modified' Gram-Schmidt:

- Orthogonalize *subsequent* rather than *preceding* vectors.
- Numerically: no difference, but sometimes algorithmically helpful.

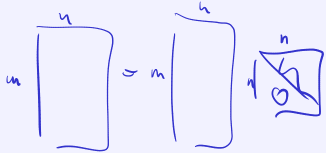
Economical/Reduced QR



Is QR with square Q for $A \in \mathbb{R}^{m\times n}$ with $m > n$ efficient?

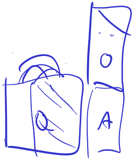
$Q^TQ = I = QQ^T$

"Economy"



$I = \underset{n\times m}{Q^T}\underset{m\times n}{Q} \sim n\times n$ ✓

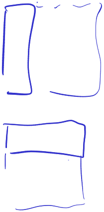
$QQ^T \neq I$



$\|Ax - b\|_2$

$= \|Q^T(QRx - b)\|$

$= \|Rx - Q^Tb\|$



Economical/Reduced QR

$= \left\| \begin{bmatrix} R \\ 0 \end{bmatrix} \vec{x} - Q^T b \right\|_2$

$(Q^T b)_{\text{bottom}}$ is not available with econ QR.

Is QR with square Q for $A \in \mathbb{R}^{m\times n}$ with $m > n$ efficient?

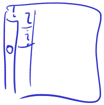
No. Can obtain economical or reduced QR with $Q \in \mathbb{R}^{m\times n}$ and $R \in \mathbb{R}^{n\times n}$. Least squares solution process works unmodified with the economical form, though the equivalence proof relies on the 'full' form.

$$\kappa_2(Q) = \|Q\|_2 \, \|Q^T\|_2$$

$$\|Q\|_2 = \max_{\|x\|_2=1} \|Qx\|_2 = \|x\| = 1$$

$$\cdots H_3 H_2 H_1 A = R$$

orthogon

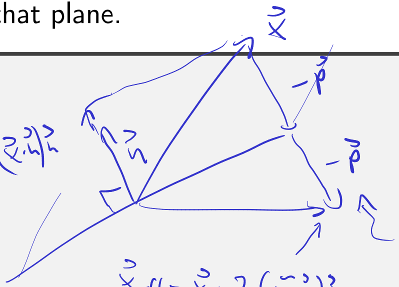


$$A = \big(\underbrace{H_1^T H_2^T H_3^T \cdots}_{Q}\big) R$$

Constructing Reflections

Given a plane represented by its (unit) normal vector $\boldsymbol{n}$, construct a reflection about that plane.

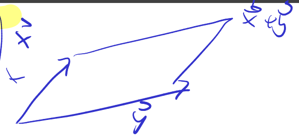
$\vec p = (\vec x \cdot \vec n)\vec n$



$\alpha \vec e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$\vec x_{refl} = \vec x - 2(\vec x \cdot \vec n)\vec n$

$= (I - 2\vec n \vec n^T)\vec x$

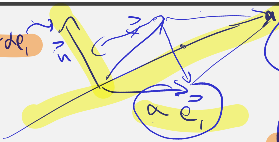


Householder Transformations

Find an *orthogonal* matrix Q to zero out the lower part of a vector $\boldsymbol{a}$.

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$ $\begin{pmatrix} ? \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\vec{u} = \vec{x} - \alpha \vec{e}_1$



$\alpha \vec{e}_1 + \vec{x}$

$$(\alpha \vec{e}_1 + \vec{x}) \cdot (\alpha \vec{e}_1 - \vec{x})$$

$$= \alpha^2 \|e_1\|^2 - \alpha \vec{e}_1 \cdot \vec{x} + x \cdot (\alpha \vec{e}_1) - \|x\|_2^2$$

$$= \|x\|_2^2 - \|x\|_2^2 = 0$$

$$\|\alpha e_1\|_2 = \|x\|_2$$

$$\pm \|\vec{x}\|_2 \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{e}_1}$$

$$H \, A = \left[\begin{array}{c|c} \alpha \vec{e}_1 & \\ \end{array} \right]$$

Householder Reflectors: Properties

Seen from picture (and easy to see with algebra):

$$H\boldsymbol{a} = \pm \|\boldsymbol{a}\|_2 \, \boldsymbol{e}_1.$$

Remarks:

- Q: What if we want to zero out only the $i+1$th through nth entry?
  A: Use $\boldsymbol{e}_i$ above.
- A product $H_n \cdots H_1 A = R$ of Householders makes it easy (and quite efficient!) to build a QR factorization.
- It turns out $\boldsymbol{v}' = \boldsymbol{a} + \|\boldsymbol{a}\|_2 \, \boldsymbol{e}_1$ works out, too–just pick whichever one causes less cancellation.
- H is symmetric
- H is orthogonal

Demo: 3x3 Householder demo [cleared]

Givens Rotations

If reflections work, can we make rotations work, too?

Demo: 3x3 Givens demo [cleared]