September 17, 2024

Announcements

- HW 1 graded
- Exam 1 live

Goals

- residual → est. error
- $\frac{\|\Delta A\|}{\|A\|}$
- orth SVD, mat. norm
- CN

Review

$\|AB\| \le \|A\| \|B\|$

→ $\|Ax\| \le \|A\| \|x\|$

submultiplicativity

rel. outputerr ≤ κ · rel. inputerr

linear solve: $A\vec{x} = \vec{b}$

sol, output

rhs, input

$\kappa(A) = \|A\| \|A^{-1}\|$

$\kappa(A) = 0.5$

$\|A\|\|A^{-1}\| \ge \|AA^{-1}\| = \|I\| = 1$

Residual Vector

What is the residual vector of solving the linear system

$$\boldsymbol{b} = A\boldsymbol{x}?$$

$$\vec{r} = \vec{b} - A\vec{x}$$

Residual and Error: Relationship

$A\vec{x}=\vec{b} \Leftrightarrow \vec{x}=A^{-1}\vec{b}$

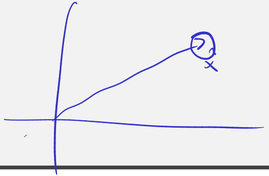
How do the (norms of the) residual vector $\boldsymbol{r}$ and the error $\Delta \boldsymbol{x} = \boldsymbol{x} - \widehat{\boldsymbol{x}}$ relate to one another?

$$\|\Delta\vec{x}\| = \|\vec{x}-\hat{\vec{x}}\| = \|A^{-1}(\vec{b}-A\hat{\vec{x}})\| = \|A^{-1}\vec{r}\|$$

$$\frac{\|\Delta \vec{x}\|}{\|\hat{\vec{x}}\|} = \frac{\|A^{-1}\vec{r}\|}{\|\hat{\vec{x}}\|} \le \frac{\|A^{-1}\|\,\|\vec{r}\|}{\|\hat{\vec{x}}\|} = \kappa(A)\frac{\|r\|}{\|A\|\,\|\hat{\vec{x}}\|} \le \kappa(A)\frac{\|\vec{r}\|}{\|A\hat{\vec{x}}\|}$$

"rel. error"

comparable



Changing the Matrix

So far, only discussed changing the RHS, i.e. $A\boldsymbol{x} = \boldsymbol{b} \quad \to \quad A\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{b}}$.
The matrix consists of FP numbers, too—it, too, is approximate. I.e.

$$A\boldsymbol{x} = \boldsymbol{b} \quad \to \quad \widehat{A}\widehat{\boldsymbol{x}} = \boldsymbol{b}.$$

What can we say about the error due to an approximate matrix?

$$\Delta \vec{x} = \hat{\vec{x}} - \vec{x} = A^{-1}(A\hat{x} - b) = A^{-1}(A\hat{x} - \hat{A}\hat{x}) = A^{-1}(A - \hat{A})\hat{x}$$

$$= A^{-1}\Delta A\hat{x}$$

$$\|\Delta\vec{x}\| = \|A^{-1}\Delta A\hat{x}\| \le \|A^{-1}\|\,\|\Delta A\|\,\|\hat{x}\|$$

$$\Leftrightarrow \quad \frac{\|\Delta x\|}{\|\hat{x}\|} \le \|A^{-1}\|\,\|A\| \cdot \frac{\|\Delta A\|}{\|A\|} = \kappa(A) \cdot \frac{\|\Delta A\|}{\|A\|}$$

"rel. error"

Changing Condition Numbers

cond(AB) ≤ cond(A) cond(B)

Once we have a matrix A in a linear system $A\boldsymbol{x} = \boldsymbol{b}$, are we stuck with its condition number? Or could we improve it?

left/right "Preconditioner"

$Ax=b$

assume ∃ M so that (MA) is well-cond. ⇔ $MAx = Mb$

(AM) ⇔ $AMy = b$

$My = x$

A typical case: use diagonal matrix as the preconditioner. What is the effect in each case?

(demo)

Recap: Orthogonal Matrices

What's an *orthogonal (=orthonormal) matrix*?

One that satisfies $Q^T Q = I$ and $QQ^T = I$.

How do orthogonal matrices interact with the 2-norm?

$$\|Q\boldsymbol{v}\|_2^2 = (Q\boldsymbol{v})^T(Q\boldsymbol{v}) = \boldsymbol{v}^T Q^T Q\boldsymbol{v} = \boldsymbol{v}^T\boldsymbol{v} = \|\boldsymbol{v}\|_2^2 .$$

Singular Value Decomposition (SVD)

What is the *Singular Value Decomposition* of an $m \times n$ matrix?

Computing the 2-Norm

Using the SVD of A, identify the 2-norm.

Express the matrix condition number $\mathrm{cond}_2(A)$ in terms of the SVD:

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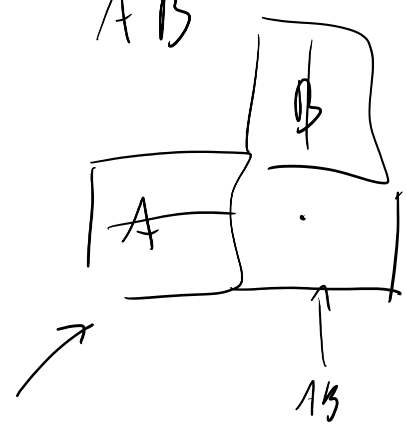
$$\underset{\text{diag}}{\overset{}{A\,M}}$$

$$\underset{\text{diag}}{M}\,A$$

$A\,B$



$(\vec{a}, \vec{b}) = \vec{a}^T\vec{b}$

$v \in \mathbb{R}^n$

$$\|Q\vec{v}\|_2^2 = (Qv,\ Qv) = (Q\vec{v})^T(Q\vec{v})$$
$$= v^T Q^T Q \vec{v} = v^T v = \|v\|_2^2$$

$$\|Q\vec{v}\|_2 = \|v\|_2$$

$$\|QA\|_2 = \|A\|_2 \qquad \|AQ\|_2 = \|A\|_2$$

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SVD

$$A = U \Sigma V^T$$

orth diag orth

$m \times n$ → $m \times n$ $n \times n$ $n \times n$ } "economy" ($m > n$)

→ $m \times m$ $m \times m$ $m \times n$ ← math ($m < n$)

→ $m \times m$ $m \times n$ $n \times n$

square square

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \cdots \sigma_n \geq 0$

$$\|A\|_2 = \|U \Sigma V^T\|_2 = \|\Sigma V^T\|_2 = \|\Sigma\|_2 = \sigma_1$$

$$= \max_i \sigma_i$$

$\kappa_2(A)$?