October 10, 2024

Announcements

- Exam 1
- HW 5

Goals

-

- Eigenvalues
  - LA review
  - Sensitivity
  - Methods

Review

Comparing the Methods

Methods to solve least squares with A an $m \times n$ matrix:

- Form: $A^T A$: $n^2 m/2$ (symmetric—only need to fill half)
  Solve with $A^T A$: $n^3/6$ (Cholesky)
- Solve with Householder: $mn^2 - n^3/3$
- If $m \approx n$, about the same
- If $m \gg n$: Householder QR requires about twice as much work as normal equations
- SVD: $mn^2 + n^3$ (with a large constant)

Demo: Relative cost of matrix factorizations [cleared]

Outline

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

- $\boldsymbol{x} \neq 0$ is called an *eigenvector* of A if there exists a λ so that

$$A\boldsymbol{x} = \lambda \boldsymbol{x}.$$

- In that case, λ is called an *eigenvalue*.
- The set of all eigenvalues $\lambda(A)$ is called the *spectrum*.
- The spectral radius is the magnitude of the biggest eigenvalue:

$$\rho(A) = \max\{|\lambda| : \lambda(A)\}$$

$\sigma(A)$

Eigenvalue Problems: Motivation from Mechanics

Consider mass-spring systems, e.g. as modeled in (e.g.) myphysicslab.com
What is needed to model?

$$\vec{F} = m\vec{a} \qquad \vec{a} = \frac{\partial^2}{\partial t^2}\vec{x} \qquad \vec{F} = \alpha(\vec{x} - \vec{x}_0)$$

$$\vec{F} = A\vec{x} \iff A\vec{x}(t) = m\vec{a} = \underbrace{m}_{=1} \cdot \frac{\partial^2 \vec{x}}{\partial t^2}$$

$$\vec{x}(t) = \sin(\omega t) \cdot \vec{x}$$

$$\cancel{\sin(\omega t)} A\vec{x} = (-\omega^2)\cancel{\sin(\omega t)}\,\vec{x}$$

Finding Eigenvalues

How do you find eigenvalues?

$\vec{x} \neq \vec{0}$

$$A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow (A - \lambda I)\mathbf{x} = 0$$
$$\Leftrightarrow A - \lambda I \text{ singular} \Leftrightarrow \det(A - \lambda I) = 0$$

$\det(A - \lambda I)$ is called the *characteristic polynomial*, which has degree n, and therefore n (potentially complex) roots.

Does that help algorithmically? Abel-Ruffini theorem: for $n \geqslant 5$ is no general formula for roots of polynomial. IOW: no.

- For LU and QR, we obtain *exact* answers (except rounding).
- For eigenvalue problems: not possible—must *iterate*.

Demo: Rounding in characteristic polynomial using SymPy [cleared]

Multiplicity

What is the *multiplicity* of an eigenvalue?

Actually, there are two notions called multiplicity:

- *Algebraic Multiplicity*: multiplicity of the root of the characteristic polynomial
- *Geometric Multiplicity*: #of lin. indep. eigenvectors

In general: AM $\geqslant$ GM.
If AM $>$ GM, the matrix is called *defective*.

An Example

Give characteristic polynomial, eigenvalues, eigenvectors of

$$\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}.$$

CP: $(1-\lambda)^2$

AM: 2 for both eigenvalues of 1

$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \iff x+y = x \iff y=0$

eigenspace spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Diagonalizability

When is a matrix called *diagonalizable*?

If it's not defective

$\Rightarrow n$ eigenvectors $\Rightarrow$ a basis of eigenvectors x_i

$$X = \begin{pmatrix} x_1 & | & \cdots & | & x_n \end{pmatrix} \qquad A\vec{x}_i = \lambda_i \vec{x}_i$$

D

$$AX = X \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$X^{-1} A X = D$$

similarity transform

Similar Matrices

$X^{-1}A = BX^{-1}$

$X^{-1}AX = B$

Related definition: Two matrices A and B are called similar if there exists an invertible matrix X so that $A = XBX^{-1}$.

In that sense: "Diagonalizable" = "Similar to a diagonal matrix".

Observe: Similar A and B have same eigenvalues. (Why?)

Suppose $A\vec{v} = \lambda\vec{v}$ $(\vec{v} \neq 0)$. $\quad \vec{w} = X^{-1}\vec{v}$

$$B\vec{w} = X^{-1}A\underbrace{XX^{-1}}_{\mathrm{id}}\vec{v} = X^{-1}A\vec{v} = \lambda X^{-1}\vec{v} = \lambda\vec{w}$$

Eigenvalue Transformations (I)

What do the following transformations of the eigenvalue problem $A\boldsymbol{x} = \lambda\boldsymbol{x}$ do?

Suppose $A\vec{v} = \lambda\vec{v}$ $(\vec{x} \neq 0)$.

Shift. $A \to A - \sigma I$

$$(A - \sigma I)\vec{v} = A\vec{v} - \sigma\vec{v} = \lambda\vec{v} - \sigma\vec{v} = (\lambda - \sigma)\vec{v}$$

Inversion. $A \to A^{-1}$

$$A\vec{x} = \lambda\vec{x} \quad | A^{-1}\cdot \quad \Leftrightarrow \quad \vec{x} = \lambda A^{-1}\vec{x} \Leftrightarrow \frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}$$

Power. $A \to A^k$

$A^3 = A \cdot A A$

$$A^k\vec{x} = A^{k-1}Ax = A^{k-1}\lambda\vec{x} = \dots = \lambda^k\vec{x}$$

Eigenvalue Transformations (II)

Polynomial $A \to aA^2 + bA + cI$

$$(aA^2 + bA + cI)\vec{x} = (a\lambda^2 + b\lambda + c)\vec{x}$$

Similarity $T^{-1}AT$ with T invertible

$$\vec{y} = T^{-1}\vec{x}$$

eigval: same
eigenvectors: basis transform

$$T^{-1}AT\vec{y} = T^{-1}A\underbrace{TT^{-1}}_{Id}x = \lambda\vec{y}.$$

Sensitivity (I)

Assume A not defective. Suppose $X^{-1}AX = D$. Perturb $A \to A + E$. What happens to the eigenvalues?

$$\underbrace{X^{-1}(\underbrace{A+E}_{})X = \underbrace{D + F}_{}}_{}$$

(F not necessarily diagonal)

same eigvals because similar

Suppose $\vec{v} \neq \vec{0}$ is eigenvector of $D+F$,

$$(D+F)\vec{v} = \mu\vec{v}$$

$$F\vec{v} = (\mu I - D)\vec{v} \quad \Big| \quad (\mu I - D)^{-1}\cdot$$

assume μ is not an eigenv. of A.

$$\vec{v} = (\mu I - D)^{-1} F \vec{v}$$

$$\underbrace{\cancel{\|\vec{v}\|}}_{1} \le \|(\mu I - D)^{-1}\| \, \|F\| \, \cancel{\|\vec{v}\|}$$

Sensitivity (II)

$X^{-1}(A+E)X = D + F$. Have $\left\|(\mu I - D)^{-1}\right\|^{-1} \le \|F\|$.

Demo: Bauer-Fike Eigenvalue Sensitivity Bound [cleared]

$\|(\mu I - D)^{-1}\|^{-1}$ is the distance between μ and the closest eigenvalue of A for μ

$$|\mu - \lambda_k| \le \|F\| = \|X^{-1} E X\| \le \kappa(X) \|E\|$$

$$F = X^{-1} E X$$