October 17, 2024

Announcements

- Exam 2 RSN
- Exam 3 res.
  - ↳ collective cheat sheet

Goals

- wrap up power it.
- Compute all eigv. / eigv.
  - → QR algorithm
  - → Schur form

Review

Power it.

$\tilde{x}_0$ random

$\tilde{\tilde{x}}_{k+1} = A \tilde{x}_k$

$\tilde{x}_{k+1} = \frac{\tilde{\tilde{x}}_{k+1}}{\|\tilde{\tilde{x}}_{k+1}\|}$

$\rightarrow \tilde{v}_1$ (largest $|\lambda|$)

$\tilde{x}_0$ random

$\tilde{\tilde{x}}_{k+1} = A^{-1} \tilde{x}_k$

$\tilde{x}_{k+1} = \frac{\tilde{\tilde{x}}_{k+1}}{\|\tilde{\tilde{x}}_{k+1}\|}$

$\rightarrow \tilde{v}_n$ (smallest $|\lambda|$)

$\tilde{x}_0$ random

$\tilde{\tilde{x}}_{k+1} = (A-\sigma I)^{-1} \tilde{x}_k$

$\tilde{x}_{k+1} = \frac{\tilde{\tilde{x}}_{k+1}}{\|\tilde{\tilde{x}}_{k+1}\|}$

$\leftarrow \tilde{v}_?$ (closest to σ)

What about Eigenvalues?

Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

$$A\vec{x} = \lambda \vec{x}$$

$$e_{k+1} \approx C \cdot e_k^2$$

$$\Rightarrow \lambda = \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}$$

Rayleigh quotient

Demo: Power Iteration and its Variants [cleared]

Schur form: Motivation

$X A X^{-1} = D$

Schur form

For finding multiple eigenvalues, want factorization that allows access to all eigenvalues and eigenvectors.
Suggestions?

$Q A Q^T = U$

eigenvalues on diagonal

Schur form: Motivation

For finding multiple eigenvalues, want factorization that allows access to all eigenvalues and eigenvectors.
Suggestions?

- Diagonalization $A = XDX^{-1}$ cannot provide what we need: it does not always exist.
- Even if it did exist, computing/applying X^{-1} would be subject to rounding concerns.
  - Idea: use a similarity transform with *orthogonal* matrices.

Schur form

Show: Every matrix is orthonormally similar to an upper triangular matrix, i.e. $A = QUQ^T$. This is called the Schur form or Schur factorization.

Suppose $(\lambda_1, \vec{v})$

$$\underbrace{\begin{pmatrix} \vec{v}_1 & \begin{matrix}\text{Basis of} \\ \{\vec{v}_1\}^\perp\end{matrix} \end{pmatrix}}_{Q_1} \begin{pmatrix} \lambda & ? \\ 0 & \\ 0 & \\ 0 & \\ 0 & \end{pmatrix} \begin{pmatrix} Q_1^T \end{pmatrix} = A$$

$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{Q_1^T \vec{v}} \qquad = A v_1$$

$$Q_1 Q_2 Q_3 \cdots Q_n \overset{U}{\boxed{\diagdown}} Q_n^T \cdots Q_2^T Q_1^T = A$$

Schur Form: Comments, Eigenvalues, Eigenvectors

$A = QUQ^T$. For complex λ:

- Either complex matrices, or
- 2×2 blocks on diag.

$Q A Q^\dagger = U$

If we had a Schur form of A (no 2×2 blocks), can we find the eigenvalues?

And the eigenvectors?

Find eigvec of U, use similarity.

$\vec{x} = (U_{11}^{-1}\vec{u}, -1, 0)^T$

$$U - \lambda I = \begin{pmatrix} U_{11} & \vec{u} & U_{13} \\ & 0 & \vec{v}^T \\ & & U_{31} \end{pmatrix}$$

$\Rightarrow (U - \lambda I)\vec{x} = 0$

$\Rightarrow Ux = \lambda I$

$\vec{y} = Q^T \vec{x}$ $\qquad$ $Q A Q^T = U$

$A = Q^T U Q$

$$A\vec{y} = Q^T U Q Q^T x = Q^T U x = Q^T \lambda x$$
$$= \lambda \vec{y}$$

$O(n^2)$ to compute an eigvec

$O(n^3)$ to compute all.

Computing Multiple Eigenvalues

All Power Iteration Methods compute one eigenvalue at a time.
What if I want *all* eigenvalues?

- Deflation
- Run n power iterations

Simultaneous Iteration

What happens if we carry out power iteration on multiple vectors simultaneously?

$$X_0 \in \mathbb{R}^{n \times n} \quad \text{or} \quad \mathbb{C}^{n \times n}$$

$$X_1 = A X_0$$

$$\vdots$$

Orthogonal Iteration

$X_0 \in \mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$

$Y_1 = A X_0$

$Q_1 R_1 = Y_1$

$X_1 = Q_1$

$Y_2 = A X_1$

$\vdots$

- Slow
- Expensive iteration

Toward the QR Algorithm

$$Q_0 R_0 = X_0$$

$$X_1 = A Q_0$$

$$Q_1 R_1 = X_1 = A Q_0 \Rightarrow A = \underbrace{Q_1 R_1 Q_0^T}$$

$$X_2 = A Q_1$$

$$Q_2 R_2 = X_2$$

$$\vdots$$

$$Q_i R_i Q_i^T \approx A$$

$$Q^T A Q \approx \nabla$$

$$\hat{X}_n := Q_n^T A Q_n \approx R_n$$

Demo: Orthogonal Iteration [cleared]

QR Iteration/QR Algorithm

QR alg.

$$\bar{X}_0 = A$$

$$\bar{Q}_k \bar{R}_k = \bar{X}_k$$

$$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k$$

$$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k = \bar{Q}_k^H \bar{X}_k \bar{Q}_k = \dots \quad \bar{Q}_k^H \dots \bar{Q}_0^H A \bar{Q}_0 \dots \bar{Q}_k$$

If R is $\triangle$, then this is Schur form!

$$A^2 = \bar{Q}_0 \bar{R}_0 \bar{Q}_0 \bar{R}_0 = \bar{Q}_0 \bar{Q}_1 \bar{R}_1 \bar{R}_0$$

$$\hat{X}_k = \bar{X}_{k+1}$$

$$Q_k = \bar{Q}_0 \dots \bar{Q}_k$$

orth. it   QR it.

$\longleftarrow$ equivalence of QR it. and orth. it.

Proof sketch: Equivalence of QR iteration/Orth. iteration

Orthogonal Iteration (no bars)

- $X_0 := A$
  - $Q_0 R_0 := X_0$,
  - where we may choose $Q_0 = \bar{Q}_0$
  - $\hat{X}_0 = Q_0^H A Q_0 = Q_0^H Q_0 R_0 Q_0 = R_0 Q_0$
- $X_1 := A Q_0$
  - $Q_1 R_1 := X_1$, and because of $X_1 = Q_0 Q_0^H A Q_0 = Q_0 \bar{X}_1 = Q_0 \bar{Q}_1 \bar{R}_1$ we may choose $Q_1 = Q_0 \bar{Q}_1 = \bar{Q}_0 \bar{Q}_1$.
- $\vdots$

QR Iteration (with bars)

- $\bar{X}_0 := A$
  - $\bar{Q}_0 \bar{R}_0 := A$
- $\bar{X}_1 := \bar{R}_0 \bar{Q}_0 = \hat{X}_0$
  - $\bar{Q}_1 \bar{R}_1 := \bar{X}_1$
- $\bar{X}_2 := \bar{R}_1 \bar{Q}_1$
  - $\bar{X}_2 = Q_1^H A Q_1 = \hat{X}_1$
- $\vdots$

Demo: QR Iteration [cleared]

QR Iteration: Forward *and* Inverse

QR iteration may be viewed as performing inverse iteration. How?