October 24, 2024

Announcements

- Exam 3
- Exam 4 res start 11/7
- Posting Materials

Goals

- QR cost
- Krylov
- Nonlinear eq.

Review

$X_0 = A$

$Q_k R_k = X_k \leftarrow O(n^3)$

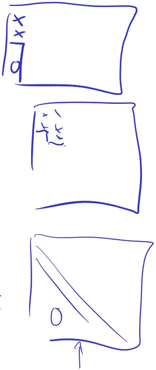
$X_{k+1} = R_k Q_k \leftarrow O(n^3)$

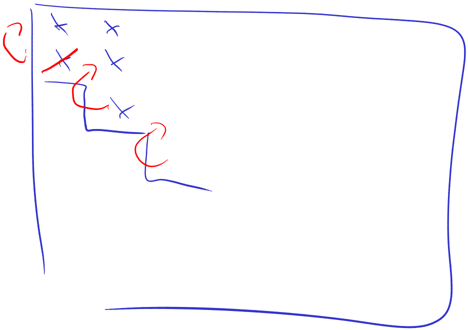
$H_1 A H_1^T =$

$H_2 H_1 A H_1^T H_2^T =$

$\vdots$

$QAQ^T =$





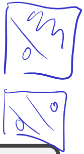
$X_0 = A$

$Q_k R_k = X_k \leftarrow \cancel{O(n^3)}$ $O(n^2)$ using Givens

$X_{k+1} = R_k Q_k \leftarrow \cancel{O(n^3)}$ $O(n^2)$

QR/Hessenberg: Overall procedure

$QAQ^T = H$



Overall procedure:

1. Reduce matrix to Hessenberg form
2. Apply QR iteration using Givens QR to obtain Schur form

Why does QR iteration *stay* in Hessenberg form?

upper Hess.

$X_0 = A$

$Q_k R_k = X_k \iff \underset{\text{Hess}}{Q_k} = \underset{\text{Hess}}{X_k}\ \underset{\text{tri}}{R_k^{-1}}$

$\underset{\text{Hess}}{X_{k+1}} = \underset{\text{tri}}{R_k}\ \underset{\text{Hess}}{Q_k}$

What does this process look like for symmetric matrices? eig eigh

Bauer Fike thm.

$|\mu - \lambda_k| \le \mathrm{cond}(X) \cdot \|E\|$

orth.

$\kappa_2(X) = 1$

$O(n)$ / iteration in QR it.

Krylov space methods: Intro

What subspaces can we use to look for eigenvectors?

orth. ir. $\operatorname{span}(A^\ell y_1, \ldots, A^\ell y_k)$

Krylov: $\vec{x}_0$

$\operatorname{span}(\vec{x}_0, A\vec{x}_0, A^2\vec{x}_0, A^3\vec{x}_0, \ldots, A^{k-1}\vec{x}_0)$

lin. indep? let's assume.

Krylov for Matrix Factorization

What matrix factorization is obtained through Krylov space methods?

$$K_k = \left(x_0, Ax_0, \dots, A^{k-1}x_0 \right) \longleftarrow \text{columns converge}$$

Fixed² post-lecture

$$AK_n = K_n \begin{pmatrix} 0 & 0 & & 0 & \\ 1 & 0 & & \vdots & \\ 0 & 1 & & & K_n^{-1}A^n x_0 \\ \vdots & 0 & & 0 & \\ 0 & \vdots & & 1 & \end{pmatrix}$$

$$K_n^{-1}AK_n = \boxed{\begin{matrix} \diagdown & \\ 0 & \diagdown \end{matrix}} \quad \text{upper Hessenberg}$$

Conditioning in Krylov Space Methods/Arnoldi Iteration (I)

What is a problem with Krylov space methods? How can we fix it?

$$Q_n R_n = K_n \iff Q_n = K_n R_n^{-1}$$

$$\underbrace{Q_n^T A Q_n}_{\text{also upper Hess.}} = (K_n R_n^{-1})^{-1} A (K_n R_n^{-1})$$

$$= R_n \underbrace{K_n^{-1} A K_n}_{\text{upper Hess}} R_n^{-1}$$

Conditioning in Krylov Space Methods/Arnoldi Iteration (II)

take q'th col.

$Q_n^T A Q_n = H$

$AQ_n = Q_n H$

$A\vec q_k = h_{1,k}\vec q_1 + \dots + h_{kk}\vec q_k + h_{k+1,k}\vec q_{k+1}$

$\vec v = A\vec q_k$

$\vec q_j^T A \vec q_k = h_{jk}$

Demo: Arnoldi Iteration [cleared] (Part 1)

Suppose I have $\vec{q}_1, \dots, \vec{q}_k$:

Let $\vec{v} = A\vec{q}_k$

$$\otimes \quad \vec{v} - \underbrace{(\vec{q}_1^{T} v)}_{h}\,\vec{q}_1 - \dots - \underbrace{(q_k^{T} v)}_{h}\,q_k = h_{k+1,k}\, q_{k+1}$$

Arnoldi Iteration

Krylov: What about eigenvalues?

How can we use Arnoldi/Lanczos to compute eigenvalues?

Demo: Arnoldi Iteration [cleared] (Part 2)

Computing the SVD (Kiddy Version)

Demo: Computing the SVD [cleared]

"Actual"/"non-kiddy" computation of the SVD:

- Bidiagonalize $A = U \begin{bmatrix} B \\ 0 \end{bmatrix} V^T$, then diagonalize via variant of QR.
- References: Chan '82 or Golub/van Loan Sec 8.6.