November 14, 2024

Announcements

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Goals

Interpolation

– $\exists$, unique, sensitivity

– wild ideas

– error theory

Review

Making the Interpolation Problem Unique

nodes

$\rightarrow (x_i, y_i) \longleftrightarrow \vec{\alpha} \qquad \varphi_1, \ldots, \varphi_n$

$$p_{n-1}(x) = \sum_{j=1}^{n} \alpha_j \varphi_j(x) \longrightarrow p_{n-1}(x_i) = y_i$$

$$p_{n-1}'(x) = \sum_{j=1}^{n} \alpha_j \varphi_j'(x)$$

$V_{ij} = \varphi_j(x_i)$ $\qquad V \vec{\alpha} = \vec{y}$

(x_1, y_1)

$(x_2, y_i') \rightsquigarrow V_{2j} = \varphi_j'(x_i)$

Prescribe $p_{n-1}(x_i)$, $p_{n-1}'(x_i)$, Hermite interpolation

Existence/Sensitivity

Solution to the interpolation problem: Existence? Uniqueness?

$\Leftrightarrow$ invertibility of V (e.g. fails if nodes repeat)

Sensitivity?

Finding α: $\kappa(V)$

$\|\vec{y}\|_\infty \le 1 \Rightarrow \|p_{n-1}\|_\infty \le ?$

$\max_{x\in[a,b]} |p_{n-1}(x)|$

Demo: Lebesgue Constant [cleared]

Modes and Nodes (aka Functions and Points)

Both function basis and point set are under our control. What do we pick?

Ideas for basis functions:

- Monomials $1, x, x^2, x^3, x^4, \ldots$
- Functions that make $V = I \to$ 'Lagrange basis'
- Functions that make V triangular $\to$ 'Newton basis'
- *Splines* (piecewise polynomials)
- *Orthogonal polynomials*
- Sines and cosines
- 'Bumps' (*'Radial Basis Functions'*)

Ideas for points:

- Equispaced
- *'Edge-Clustered'* (so-called Chebyshev/Gauss/... nodes)

Specific issues:

- Why *not* monomials on equispaced points?
  Demo: Monomial interpolation [cleared]
- Why not equispaced?
  Demo: Choice of Nodes for Polynomial Interpolation [cleared]

Lagrange Interpolation

Find a basis so that $V = I$, i.e.

$$\varphi_j(x_i) = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

$x_1 \quad x_2 \quad x_3$

$$\varphi_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} \Rightarrow \begin{array}{l} \varphi_1(x_1) = 1 \\ \varphi_1(x_2) = 0 \\ \varphi_1(x_3) = 0 \end{array}$$

$$\varphi_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

Lagrange Polynomials: General Form

$$\varphi_j(x) = \frac{\prod_{k=1,k\neq j}^{m}(x - x_k)}{\prod_{k=1,k\neq j}^{m}(x_j - x_k)}$$

Write down the Lagrange interpolant for nodes $(x_i)_{i=1}^m$ and values $(y_i)_{i=1}^m$.

$$p_{n-1}(x) = \sum_{i=1}^{n} y_i \varphi_i(x)$$

barycentric interpolation → look up!

Newton Interpolation

Find a basis so that V is triangular.

$$\varphi_j(x) = \prod_{k=1}^{j} (x - x_k)$$

$\to V$ triangular

"divided differences" is an alternative to Fw/ bw subst. for finding $\vec{\alpha}$, also at $O(n^2)$ cost

Why not Lagrange/Newton?

hard to do calculus on.

Newton Interpolation

Find a basis so that V is triangular.

Why not Lagrange/Newton?

Cheap to form, expensive to evaluate, expensive to do calculus on.

Better conditioning: Orthogonal polynomials

What caused monomials to have a terribly conditioned Vandermonde?

What's a way to make sure two vectors are *not* like that?

But polynomials are functions!

Better conditioning: Orthogonal polynomials

What caused monomials to have a terribly conditioned Vandermonde?

Being close to linearly dependent.

What's a way to make sure two vectors are *not* like that?

Orthogonality

But polynomials are functions!

Orthogonality of Functions

How can functions be orthogonal?

$(\vec{f}, \vec{g}) = \quad \vec{f} \cdot \vec{g} = \sum f_i \cdot g_i \quad \leftarrow$

$(f, g) = \quad = \int_{-1}^{1} f(x)\, g(x)\, dx$

$\psi : [-1, 1] \longrightarrow [a, b] \qquad \psi(r) = \left(\frac{r}{2} + 1\right) \cdot (b - a) + a$

ψ^{-1}

Constructing Orthogonal Polynomials

How can we find an orthogonal basis?

Demo: Orthogonal Polynomials [cleared] — Got: Legendre polynomials.

But how can I practically compute the Legendre polynomials?

Chebyshev Polynomials: Definitions

$(f, g)_L = \int_{-1}^{1} f \cdot g \, dx$

$(f, g)_C = \int_{-1}^{1} f g \frac{1}{\sqrt{1-x^2}}$

Three equivalent definitions:

- Result of Gram-Schmidt with weight $1/\sqrt{1-x^2}$. What is that weight?

  1 / half-circle

  (Like for Legendre, you won't exactly get the standard normalization if you do this.)

- $T_k(x) = \cos(k \cos^{-1}(x))$
- $T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$ plus $T_0 = 1,\ T_1 = x$

recurrence, efficient to eval.

Chebyshev Interpolation

What is the Vandermonde matrix for Chebyshev polynomials?