November 19, 2024

Announcements

- Exam 4

Goals

$[-1,1]$

$$T_k(x) = \cos(k \cos^{-1}(x))$$

$\in [-1,1]$

Review

- Monomials / equispaced
  - conditioning bad → orthogonal
  - Lebesgue constant

$$\Lambda = \max_{\vec{y} \neq 0} \frac{\|p_{n-1}\|_\infty}{\|\vec{y}\|_\infty} \quad \begin{array}{l} \leftarrow \text{function inf norm} \\ \leftarrow \text{vector inf norm} \end{array}$$

$$(f,g) = \int_{-1}^{1} f(x)g(x)\,w(x)\,dx$$

$$(f,g) = 0$$

Chebyshev Interpolation

$T_k(x) = \cos(k \cos^{-1}(x))$

What is the Vandermonde matrix for Chebyshev polynomials?

$$x_i = \cos\left(\frac{i}{k}\pi\right) \qquad (i = 0, \dots, k)$$

$$V_{ij} = T_j(x_i) = \cos\left(j \cancel{\cos^{-1}}\left(\cos\left(\frac{i\pi}{k}\right)\right)\right)$$

$$= \cos\left(\frac{j\, i\, \pi}{k}\right)$$

DCT

Using FFT, multiply, inverse, apply } in $O(n \log n)$

$$V\alpha = \vec{y}$$

Chebyshev Nodes

Might also consider roots (instead of extrema) of T_k:

$$x_i = \cos\left(\frac{2i-1}{2k}\pi\right) \quad (i = 1\dots, k).$$

Vandermonde for these (with T_k) can be applied in $O(N \log N)$ time, too.

Edge-clustering seemed like a good thing in interpolation nodes. Do these do that?

Demo: Chebyshev Interpolation [cleared] (Part I-IV)

Chebyshev Interpolation: Summary

- Chebyshev interpolation is fast and works extremely well
- http://www.chebfun.org/ and: ATAP
- In 1D, they're a very good answer to the interpolation question
- But sometimes a piecewise approximation (with a specifiable level of smoothness) is more suited to the application

Truncation Error in Interpolation

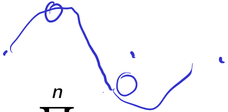
If f is n times continuously differentiable on a closed interval I and $p_{n-1}(x)$ is a polynomial of degree at most $\cancel{n}$ $n-1$ that interpolates f at n distinct points $\{x_i\}$ $(i = 1, ..., n)$ in that interval, then for each x in the interval there exists ξ in that interval such that

$$R(x) = f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi(x))}{n!}\underbrace{(x - x_1)(x - x_2)\cdots(x - x_n)}_{W(x)}.$$

$$R(x) = f(x) - p_{n-1}(x)$$

$$Y_x(t) = R(t) - \frac{R(x)}{W(x)}W(t) \qquad W(x) = \prod_{i=1}^{n}(x - x_i)$$

Truncation Error in Interpolation: cont'd.



$$Y_x(t) = R(t) - \frac{R(x)}{W(x)} W(t) \quad \text{where} \quad W(t) = \prod_{i=1}^{n} (t - x_i)$$

$R = f - p_{n-1}$

- Y_x has $n+1$ roots: $x_1, \dots, x_n, x$
- Y_x' has n roots
- $Y_x^{(n)}$ has 1 root: $\xi(x)$

$$Y_x^{(n)}(t) = R^{(n)}(t) - \frac{R(x)}{W(x)} n!$$

$$= f^{(n)}(t) - \frac{R(x)}{W(x)} n!$$

$$0 = Y^{(n)}(\xi(x)) = f^{(n)}(\xi(x)) - \frac{R(x)}{W(x)} n!$$

Error Result: Connection to Chebyshev

What is the connection between the error result and Chebyshev interpolation?

$W(x)$ nicely bounded for Cheb nodes

Demo: Chebyshev Interpolation [cleared] (Part V)

Error Result: Connection to Chebyshev

What is the connection between the error result and Chebyshev interpolation?

- The error bound suggests choosing the interpolation nodes such that the product $|\prod_{i=1}^{n}(x - x_i)|$ is as small as possible. The Chebyshev nodes achieve this.
- If nodes are edge-clustered, $\prod_{i=1}^{n}(x - x_i)$ clamps down the (otherwise quickly-growing) error there.
- Confusing: *Chebyshev approximating polynomial* (or "polynomial best-approximation"). Not the Chebyshev interpolant.
- Chebyshev nodes also do not minimize the Lebesgue constant.

Demo: Chebyshev Interpolation [cleared] (Part V)

Error Result: Simplified Form

Boil the error result down to a simpler form. $I = (a, b)$

- Assume $x_1 < \cdots < x_n$
- Assume $|f^{(n)}(x)| \le M$ for $x \in I$.
- Let $h = |I| \Rightarrow |x - x_i| \le h$

$$\max_{x \in I} |f(x) - p_{n-1}(x)| \le C \cdot M \cdot h^n$$

n-th order convergence

- **Demo: Interpolation Error [cleared]**
- **Demo: Jump with Chebyshev Nodes [cleared]**

Going piecewise: Simplest Case

Construct a piecewise linear interpolant at four points.

x_0, y_0		x_1, y_1		x_2, y_2		x_3, y_3
	$f_1 = a_1 x + b_1$		$f_2 = a_2 x + b_2$		$f_3 = a_3 x + b_3$	
	2 unk.		2 unk.		2 unk.	
	$f_1(x_0) = y_0$		$f_2(x_1) = y_1$		$f_3(x_2) = y_2$	
	$f_1(x_1) = y_1$		$f_2(x_2) = y_2$		$f_3(x_3) = y_3$	
	2 eqn.		2 eqn.		2 eqn.	

Why three intervals?