September 12, 2024

Announcements

HW 2 due

Exam 1

Goals

- norms
- conditioning of $Ax=b$
- "cond. number of a matrix"
- uses thereof
- SVD

Review

Solving a Linear System

Given:

- $m \times n$ matrix A
- m-vector $\boldsymbol{b}$

out in

$Ax = b$

What are we looking for here, and when are we allowed to ask the question?

Next: Want to talk about conditioning of this operation. Need to measure distances of matrices.

Solving a Linear System

rank-nullity-theorem

rank (A) + dim $N(A)$ = # col of A

↑ "full" rank: as big as it can get

Given:

- $m \times n$ matrix A
- m-vector $\boldsymbol{b}$

What are we looking for here, and when are we allowed to ask the question?

Want: n-vector $\boldsymbol{x}$ so that $A\boldsymbol{x} = \boldsymbol{b}$.

- Linear combination of columns of A to yield $\boldsymbol{b}$.
- Restrict to square case $(m = n)$ for now.
- Even with that: solution may not exist, or may not be unique.

Unique solution exists iff A is *nonsingular*.

Next: Want to talk about conditioning of this operation. Need to measure distances of matrices.

Matrix Norms

$A\vec{x} = \lambda\vec{x}$

What norms would we apply to matrices?

$A(\alpha x) = \alpha(Ax)$

$A(x+y) = Ax + Ay$

$$* \quad \|A\vec{x}\|_{1,2,\infty} \leq \text{number} \cdot \|x\|_{1,2,\infty}$$

the smallest number so that $\circledast$ holds is called $\|A\|$. For all x

$\|A\|$ → just a number

For all x: $\quad \|Ax\| \leq \|A\| \, \|x\|$

$$\Leftrightarrow \frac{\|Ax\|}{\|x\|} \leq \|A\| \quad \Leftrightarrow \quad \|A \underbrace{\frac{x}{\|x\|}}_{\|\cdot\| = 1}\| \leq \|A\|$$

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

Identifying Matrix Norms

What is $\|A\|_1$? $\|A\|_\infty$?

$n \left(\right)$ 1

$$\|A\|_1 = \max_{\text{col } j} \sum_{\text{row } i} |A_{ij}| \qquad \|A\|_\infty = \max_{\text{row } j} \sum_{\text{col } i} |A_{ij}|$$

How do matrix and vector norms relate for $n \times 1$ matrices?

$A = \begin{pmatrix} \vec{x} \end{pmatrix}$

$\|A\| = \|x\|$

matrix norm

vector norm

Demo: Matrix norms [cleared]

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

- $\|A\| > 0 \Leftrightarrow A \neq 0$.
- $\|\gamma A\| = |\gamma| \, \|A\|$ for all scalars γ.
- Obeys triangle inequality $\|A + B\| \leq \|A\| + \|B\|$

But also some more properties that stem from our definition:

$$\|Ax\| \leq \|A\| \, \|x\|$$ ← sub-multiplicativity

$$\|AB\| \leq \|A\| \, \|B\|$$

In these notes: If we write $\|\cdot\|$ (for matrix norms) without any specifics, then the statement is true for any induced norm. If a specific norm is needed, the notation will indicate that.

$\|\cdot\|_2$

Conditioning

What is the condition number of solving a linear system $Ax = b$?

outp. in

$\Delta\vec{x}$ $\Delta\vec{b}$

~~max~~ $\dfrac{\text{rel. err. in output}}{\text{rel. err. in input}} = \dfrac{\|\Delta x\| / \|x\|}{\|\Delta b\| / \|b\|} = \dfrac{\|\Delta x\|\ \|b\|}{\|\Delta b\|\ \|x\|}$

$= \dfrac{\|A^{-1}\Delta b\|\ \|Ax\|}{\|\Delta b\|\ \|x\|} \le \|A^{-1}\| \dfrac{\|\Delta b\|}{\|\Delta b\|} \cdot \|A\| \cdot \dfrac{\|x\|}{\|x\|}$

$= \|A^{-1}\| \cdot \|A\|$

does not depend on $\vec{b}$.

$\|Ax\| \le \|A\|\ \|b\|$

$Ax = b$

$A(x + \Delta x) = b + \Delta b$

$A\Delta x = \Delta b \iff \Delta x = A^{-1}\Delta b$

Conditioning of Linear Systems: Observations

Showed $\kappa(\text{Solve } A\boldsymbol{x} = \boldsymbol{b}) \le \left\|A^{-1}\right\| \|A\|$.
I.e. found an *upper bound* on the condition number. With a little bit of fiddling, it's not too hard to find examples that achieve this bound, i.e. that it is *sharp*.

So we've found the *condition number of linear system solving*, also called the condition number of the matrix A:

$$\operatorname{cond}(A) = \kappa(A) = \|A\| \left\|A^{-1}\right\|.$$

Conditioning of Linear Systems: More properties

- cond is relative to a given norm. So, to be precise, use

$$\text{cond}_2 \quad \text{or} \quad \text{cond}_\infty .$$

out in

$A x = b$

- If A^{-1} does not exist: $\text{cond}(A) = \infty$ by convention.

What is $\kappa(A^{-1})$?

$$\kappa(A^{-1}) = \kappa(A)$$

What is the condition number of matrix-vector multiplication?

in out

$Ax = b$

in out

$\Leftrightarrow x = A^{-1} b$

$$\kappa(A)$$

Demo: Condition number visualized [cleared]

Demo: Conditioning of 2x2 Matrices [cleared]

Residual Vector

What is the residual vector of solving the linear system

$$\boldsymbol{b} = A\boldsymbol{x}?$$