

September 12, 2024

Announcements

HW 2 due

Exam 1

Review

Goals

- norms
- conditioning of $Ax=b$
- "cond. number of a matrix"
- uses thereof
- SVD

Solving a Linear System

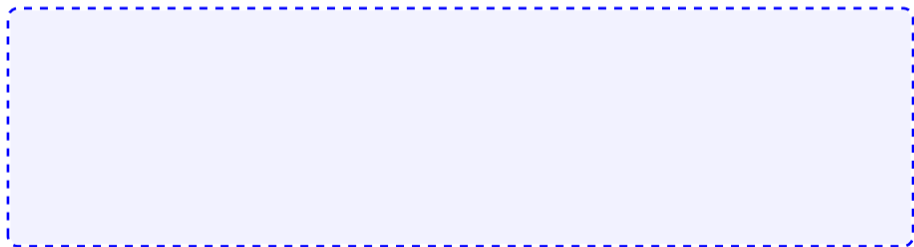
Given:

- ▶ $m \times n$ matrix A
- ▶ m -vector \mathbf{b}

$$A \mathbf{x} = \mathbf{b}$$

Handwritten diagram showing the equation $A \mathbf{x} = \mathbf{b}$. An arrow labeled "out" points from A to \mathbf{x} . An arrow labeled "in" points from \mathbf{x} to \mathbf{b} .

What are we looking for here, and when are we allowed to ask the question?



Next: Want to talk about conditioning of this operation. Need to measure distances of matrices.

Solving a Linear System

Given:

- ▶ $m \times n$ matrix A
- ▶ m -vector \mathbf{b}

What are we looking for here, and when are we allowed to ask the question?

rank-nullity theorem

$$\text{rank}(A) + \dim \mathcal{N}(A) = \# \text{ col of } A$$

↑ "full" rank: as big as sit^A can get

Want: n -vector \mathbf{x} so that $A\mathbf{x} = \mathbf{b}$.

- ▶ Linear combination of columns of A to yield \mathbf{b} .
- ▶ **Restrict** to square case ($m = n$) for now.
- ▶ Even with that: solution may not exist, or may not be unique.

Unique solution exists iff A is nonsingular.

Next: Want to talk about conditioning of this operation. Need to measure distances of matrices.

Matrix Norms

$$\cancel{A \vec{x} = \lambda \vec{x}}$$

$$A(\alpha x) = \alpha (Ax)$$

$$A(x+y) = Ax + Ay$$

What norms would we apply to matrices?

$$* \quad \|A\vec{x}\|_{\substack{1 \\ 2 \\ \infty}} \leq \text{number} \cdot \|\vec{x}\|_{\substack{1 \\ 2 \\ \infty}}$$

the smallest number so that $(*)$ holds is called $\|A\|$. For all x

For all x : $\|Ax\| \leq \|A\| \|x\|$

$$\Leftrightarrow \frac{\|Ax\|}{\|x\|} \leq \|A\| \quad \Leftrightarrow \|A \underbrace{\frac{x}{\|x\|}}_{\|\cdot\|=1}\| \leq \|A\|$$

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

Identifying Matrix Norms

What is $\|A\|_1$? $\|A\|_\infty$?

$$\begin{matrix} n \\ \left(\right) \\ 1 \end{matrix}$$

$$\|A\|_1 = \max_{\text{col } j} \sum_{\text{row } i} |A_{ij}| \quad / \quad \|A\|_\infty = \max_{\text{row } j} \sum_{\text{col } i} |A_{ij}|$$

How do matrix and vector norms relate for $n \times 1$ matrices?

$$A = \begin{pmatrix} x \\ \vdots \end{pmatrix}$$

$$\|A\| = \|x\|$$

↑ matrix norm ↙ vector norm

Demo: Matrix norms [cleared]

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

- ▶ $\|A\| > 0 \Leftrightarrow A \neq 0$.
- ▶ $\|\gamma A\| = |\gamma| \|A\|$ for all scalars γ .
- ▶ Obeys triangle inequality $\|A + B\| \leq \|A\| + \|B\|$

But also some more properties that stem from our definition:

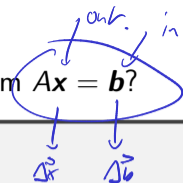
$$\|A \times x\| \leq \|A\| \|x\| \quad \leftarrow \text{sub-multiplicativity}$$
$$\|AB\| \leq \|A\| \|B\|$$

In these notes: If we write $\|\cdot\|$ (for matrix norms) without any specifics, then the statement is true for any induced norm. If a specific norm is needed, the notation will indicate that.

$$\|\cdot\|_2$$

Conditioning

What is the condition number of solving a linear system $Ax = b$?



~~unit~~

$$\frac{\text{rel. err. in output}}{\text{rel. err. in input}} =$$

$$\frac{\|\Delta x\| / \|x\|}{\|\Delta b\| / \|b\|} = \frac{\|\Delta x\| \|b\|}{\|\Delta b\| \|x\|}$$

$$= \frac{\|A^{-1} \Delta b\| \|Ax\|}{\|\Delta b\| \|x\|} \leq \|A^{-1}\| \frac{\|\Delta b\|}{\|\Delta b\|} \cdot \|A\| \cdot \frac{\|b\|}{\|x\|}$$

$$Ax = b$$

$$A(x + \Delta x) = b + \Delta b$$

$$A \Delta x = \Delta b \Leftrightarrow \Delta x = A^{-1} \Delta b$$

$$= \|A^{-1}\| \cdot \|A\|$$

o o o

does not depend on b .

$$\|Ax\| \leq \|A\| \|b\|$$

Conditioning of Linear Systems: Observations

Showed $\kappa(\text{Solve } A\mathbf{x} = \mathbf{b}) \leq \|A^{-1}\| \|A\|$.

I.e. found an *upper bound* on the condition number. With a little bit of fiddling, it's not too hard to find examples that achieve this bound, i.e. that it is *sharp*.

So we've found the *condition number of linear system solving*, also called the **condition number of the matrix A** :

$$\text{cond}(A) = \kappa(A) = \|A\| \|A^{-1}\|.$$

Conditioning of Linear Systems: More properties

- ▶ cond is relative to a given norm. So, to be precise, use

cond_2 or cond_∞ .

$$A x = b$$

Handwritten diagram: $A x = b$ with arrows pointing from x to b labeled "out" and from b to x labeled "in".

- ▶ If A^{-1} does not exist: $\text{cond}(A) = \infty$ by convention.

What is $\kappa(A^{-1})$?

$$\kappa(A^{-1}) = \kappa(A)$$

What is the condition number of matrix-vector multiplication?

$$\kappa(A)$$

$$A x = b$$

Handwritten diagram: $A x = b$ with arrows pointing from x to b labeled "out" and from b to x labeled "in". Below it, $\Leftrightarrow x = A^{-1} b$ with arrows pointing from b to x labeled "out" and from x to b labeled "in".

Demo: Condition number visualized [cleared]

Demo: Conditioning of 2x2 Matrices [cleared]

Residual Vector

What is the **residual vector** of solving the linear system

$$\mathbf{b} = A\mathbf{x}?$$

