October 8, 2024

Announcements

- Exam 2

Goals

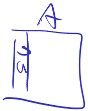
- Householder reflector
- Givens rotations
- rank-deficient
  ↳ SVD
- eigenvalues

Review

$$Q_n \cdots Q_2 Q_1 A \longrightarrow R$$

$$A = \underbrace{(Q_1^T \quad \cdots \quad Q_n^T)}_{Q} R$$

$$\kappa_2(Q) = 1$$



$$Q_1 \vec{a} = \pm \|a\|_2 \vec{e}_1$$

Householder Reflectors: Properties

Seen from picture (and easy to see with algebra):

$$H\boldsymbol{a} = \pm \|\boldsymbol{a}\|_2 \, \boldsymbol{e}_1.$$

Remarks:

- Q: What if we want to zero out only the $i+1$th through nth entry?
  A: Use $\boldsymbol{e}_i$ above.
- It turns out $\boldsymbol{v}' = \boldsymbol{a} + \|\boldsymbol{a}\|_2 \, \boldsymbol{e}_1$ works out, too–just pick whichever one causes less cancellation.
- H is symmetric
- H is orthogonal

Demo: 3x3 Householder demo [cleared] **Demo:** Householder in 3D [cleared]

Givens Rotations

If reflections work, can we make rotations work, too?

Demo: 3x3 Givens demo [cleared]

Givens Rotations: Elimination Order

Given a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

in what order can we apply Givens rotations to eliminate the nonzeros below the diagonal?

$G_{21} =$

$G_{31} =$

$\tilde{G}_{32}\tilde{G}_{21}\tilde{G}_{31}A$

$\tilde{G}_{32}\tilde{G}_{31}(\tilde{G}_{21}A)$

$a_1 \quad a_2 \quad a_3$

$\operatorname{span}(\vec{a}_1) = \operatorname{span}(\vec{q}_1)$

$\operatorname{span}(\vec{a}_1, \vec{a}_2) = \operatorname{span}(\vec{q}_1, \vec{q}_2)$

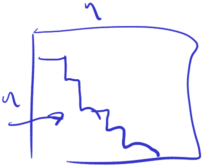
$\vdots$

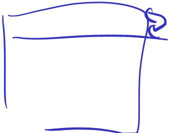
$A =$ 

One Givens

- on a vec: $O(1)$
- on a mat: $O(n)$

$O(n^2)$ rotations to get QR: $O(n^3)$





Rank-Deficient Matrices and QR

What happens with QR for rank-deficient matrices?

- 0 on diagonal of R
- Rank-revealing QR / column-pivoted QR

$$AP = QR$$

Rank-Deficient Matrices and QR

What happens with QR for rank-deficient matrices?

$A = QR$, where R has some zero diagonal entries, in undetermined order.

Practically, it makes sense to ask for all these 'small' columns to be gathered near the 'right' of R → Column pivoting.

Q: What does the resulting factorization look like?

$$AP = QR$$

$$AP = Q \begin{bmatrix} * & * & * \\ & \text{(small)} & \text{(small)} \\ & & \text{(smaller)} \end{bmatrix}$$

Also used as the basis for *rank-revealing QR*.

Rank-Deficient Matrices and Least-Squares

What happens with Least Squares for rank-deficient matrices?

$$A\boldsymbol{x} \cong \boldsymbol{b}$$

rank nullity: $\#\text{cols} = \text{rank} + \dim N(A)$

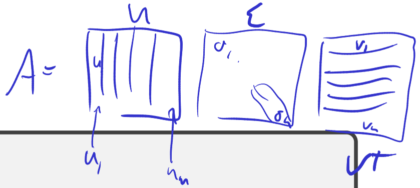
$\Rightarrow \exists \vec{n} \neq 0: \quad A\vec{n} = \vec{0}$

$$\|A\tilde{x} - b\|_2 = \min = \|A(\tilde{x} + \alpha n) - b\|_2$$

In rank-deficient LSQ:

- Ask for $\min \|Ax - b\|_2$
- Ask for $\min \|x\|_2$

SVD: What's this thing good for? (I)



$$\|A\|_2 = \sigma_1$$

$$\mathrm{cond}_2(A) = \sigma_1/\sigma_n$$

$$N(A) = \mathrm{span}\{\vec{v}_i : \sigma_i = 0\}$$

$$\mathrm{rank}(A) = \#\{\sigma_i \neq 0\}$$

$$\mathrm{numrank}(A, \varepsilon) = \#\{\sigma_i \geq \varepsilon\}$$

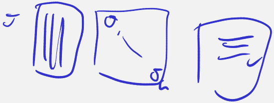
SVD: What's this thing good for? (II)

- *Low-rank Approximation*

Theorem (Eckart-Young-Mirsky)

$A = U \Sigma V^T$

If $k < r = \operatorname{rank}(A)$ *and*



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$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T, \quad \textit{then}$$

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1},$$

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_F = \|A - A_k\|_F = \sqrt{\sum_{j=k+1}^{n} \sigma_j^2}.$$

Demo: Image compression [cleared]

SVD: What's this thing good for? (III)

- The minimum norm solution to $A\boldsymbol{x} \cong \boldsymbol{b}$:

$$A = U \Sigma V^T$$

min!

$$\|r\|_2 = \|Ax - b\|_2 = \|U \Sigma V^T x - b\|_2$$

$$= \|U^T (U \Sigma V^T x - b)\|_2$$

$$= \|\underbrace{\Sigma V^T x}_{y} - U^T b\|_2$$

$$= \|\Sigma y - U^T b\|_2$$

$$\begin{array}{cc} & n \\ m & \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ & & 0 & & & \end{pmatrix} \end{array} y \cong \begin{pmatrix} U^T b \end{pmatrix}$$

$y_i = (U^T b)_i / \sigma_i \quad i \in \{1..k\}$

$y_i = 0 \quad i \geq k+1$

SVD: What's this thing good for? (III)

- The minimum norm solution to $A\boldsymbol{x} \cong \boldsymbol{b}$:

$\vec{y}$ is min. norm solution of $\Sigma \vec{y} \cong U^T \vec{b}$.

$V^T \vec{x} = \vec{y}$

$\vec{x} = V \vec{y} \Rightarrow \|\vec{x}\|_2 = \|\vec{y}\|_2$

$\vec{x}$ is the min-norm solution of $A\vec{x} \cong \vec{b}$.

$\|Qx\|_2 = \|x\|_2$

SVD: Minimum-Norm, Pseudoinverse

$A = U\Sigma V^T$

$A^{-1} = V\Sigma^{-1}U^T$

What is the minimum 2-norm solution to $A\boldsymbol{x} \cong \boldsymbol{b}$ and why?

$$\vec{x} = A^+\vec{b}$$

$$A^+ = V\Sigma^+U^T \qquad \Sigma^+ = \operatorname{diag}\begin{pmatrix} 1/\sigma_i & \sigma_i \neq 0 \\ 0 & \sigma_i = 0 \end{pmatrix}$$

$$A^+\vec{b} = V\Sigma^+U^T\vec{b}$$

same vector that solved red LSQ

Generalize the pseudoinverse to the case of a rank-deficient matrix.

$$\underset{m \times n}{A} = \underset{m \times m}{Q}\;\underset{m}{\Sigma}\;\underset{n \times n}{V^T}$$

Comparing the Methods

Methods to solve least squares with A an $m \times n$ matrix:

Demo: Relative cost of matrix factorizations [cleared]