October 15, 2024

## Announcements

- HW5

## Goals

- Computing values
  - Power method
- Computing all eigenvalues

eig    eigh

## Review

$$X = (v_1 \dots v_n)$$

$$X A X^{-1} = D$$

$$\max |\lambda_{perturbed} - \lambda_{closest\ true}| \le \kappa(X) \|E\|$$

orth. matrices have great condition

$$\kappa_2(Q) = 1$$

symm.

$A = A^T \Rightarrow X$ orthogonal

"normal" $AA^T = A^T A$

$$X A X^{-1} = \begin{pmatrix} \lambda_1 & \\ 0 & \lambda_2 \end{pmatrix} \qquad X = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}$$

$$\vec{x} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$$

$$A \vec{x} = \alpha_1 \lambda_1 \vec{v}_1 + \alpha_2 \lambda_2 \vec{v}_2$$

$$A^{1000} \vec{x} = \alpha_1 \lambda_1^{1000} + \underbrace{\alpha \lambda_2^{1000} \vec{v}_2}_{\text{super tiny}}$$

# Power Iteration

**Demo:** Motivating Power Iteration [cleared]

Let $A \in \mathbb{R}^{n\times n}$ and $A\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j$ ($j \in \{1, 2, \dots, n\}$) and $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$.

Pick some $\boldsymbol{x}_0$, consider $\boldsymbol{x}_{i+1} = A\boldsymbol{x}_i$ ($i \in \{0, \dots\}$). Called Power Iteration.

$$\vec{x}_0 = \sum_{j=1}^{n} \alpha_j \vec{v}_j \qquad \vec{x}_i = A^i \vec{x}_0 = \sum_{j=1}^{n} \alpha_j \lambda_j^i \vec{v}_j$$

$$\vec{e}_i = \frac{\vec{x}_i}{\lambda_1^i} - \alpha_1 \vec{v}_1$$

$$\|\vec{e}_{i+1}\| = \left\| \frac{\vec{x}_{i+1}}{\lambda_1^{i+1}} - \alpha_1 \vec{v}_1 \right\| = \left\| \sum_{j=1}^{n} \alpha_j \frac{\lambda_j^{i+1}}{\lambda_1^{i+1}} \vec{v}_j - \alpha_1 \vec{v}_1 \right\|$$

$$= \left\| \sum_{j=2}^{n} \alpha_j \underbrace{\left(\frac{\lambda_j}{\lambda_1}\right)}_{<1}^{i+1} \vec{v}_j \right\| \le \left(\frac{|\lambda_2|}{|\lambda_1|}\right)^{i+1} \left\| \sum_{j=2}^{n} \alpha_j \vec{v}_j \right\|$$

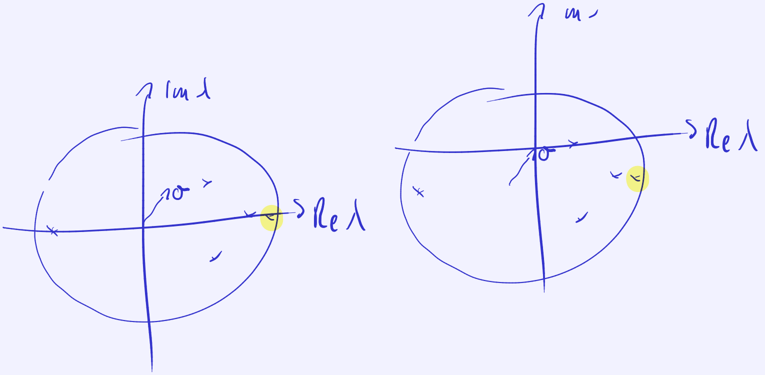
# Convergence of Power Iteration: Notation

- $\lambda_{\max}(A)$: biggest eigenvalue by magnitude
- $\lambda_{\max 2}(A)$: second-biggest eigenvalue by magnitude.
- $\lambda_{\min 2}(A)$: second-smallest eigenvalue by magnitude
- $\lambda_{\min}(A)$: smallest eigenvalue by magnitude

(Not well-defined if there are multiple $\lambda$ with the same magnitudes. Assume that's not the case.)

# Power Iteration: Shift

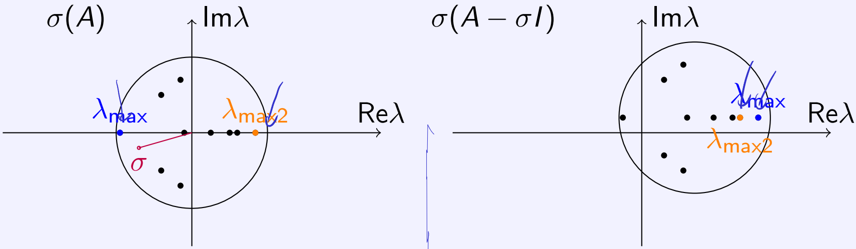
How does a shift $(A - \sigma I)$ change power iteration?



# Power Iteration: Shift

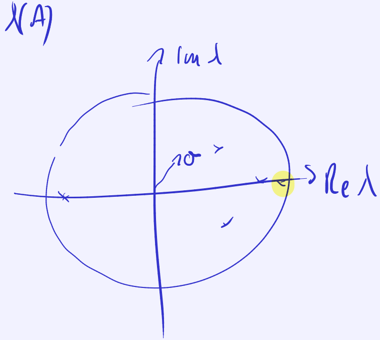
How does a shift $(A - \sigma I)$ change power iteration?

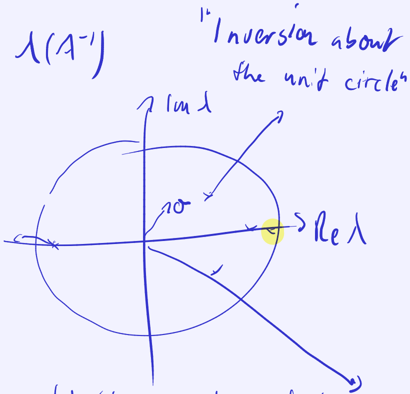
- Converges to eigenvector for $\lambda_{\max}(A - \sigma I)$ with convergence factor $\left|\frac{\lambda_{\max 2}(A-\sigma I)}{\lambda_{\max}(A-\sigma I)}\right|$.
- Can help guide convergence to eigenvalues 'on boundary' of spectrum.



# Power Iteration: Inversion

How does inversion ($A^{-1}$) change power iteration?




$$\left|\frac{\lambda_{max2}(A^{-1})}{\lambda_{max}(A^{-1})}\right| = \left|\frac{1/\lambda_{min2}(A)}{1/\lambda_{min}(A)}\right| = \left|\frac{\lambda_{min}(A)}{\lambda_{min2}(A)}\right|$$

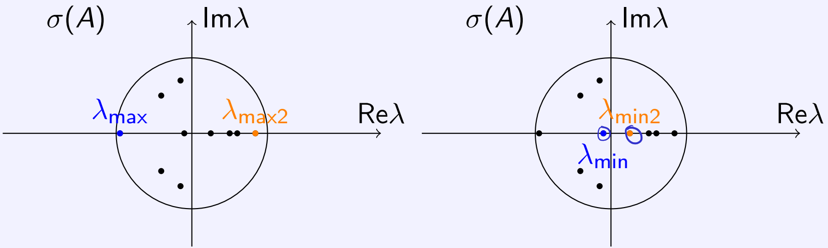
# Power Iteration: Inversion

How does inversion ($A^{-1}$) change power iteration?

- Converges to eigenvector for $\lambda_{\max}(A^{-1}) = 1/\lambda_{\min}(A)$ with convergence factor

$$\left|\frac{\lambda_{\max 2}(A^{-1})}{\lambda_{\max}(A^{-1})}\right| = \left|\frac{1/\lambda_{\min 2}(A)}{1/\lambda_{\min}(A)}\right| = \left|\frac{\lambda_{\min}(A)}{\lambda_{\min 2}(A)}\right|.$$

- Guide convergence to smallest eigenvalues.



# Power Iteration: Shift and Inversion

How does shift-invert ($(A - \sigma I)^{-1}$) change power iteration?

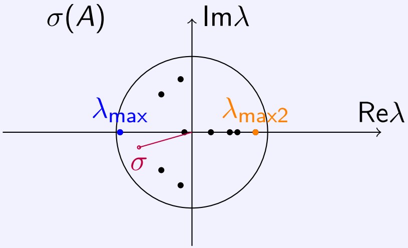
# Power Iteration: Shift and Inversion

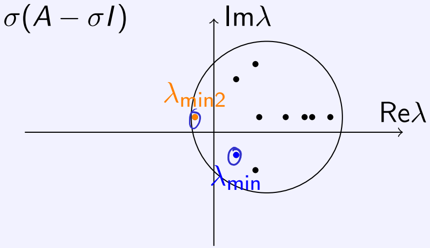
How does shift-invert ($(A - \sigma I)^{-1}$) change power iteration?

- Converges to eigenvector for $\lambda_{\max}((A - \sigma I)^{-1}) = 1/\lambda_{\min}(A - \sigma I)$ with convergence factor

$$\left|\frac{\lambda_{\max 2}((A - \sigma I)^{-1})}{\lambda_{\max}((A - \sigma I)^{-1})}\right| = \left|\frac{\lambda_{\min}(A - \sigma I)}{\lambda_{\min 2}(A - \sigma I)}\right|.$$

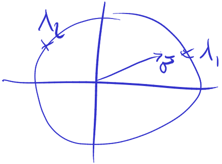
- Guide convergence to eigenvalue closest to $\sigma$.



# What could go wrong with power iteration?

- $|\lambda_1| = |\lambda_2|$ → $\lambda_1 = \lambda_2$ multiplicity → ?
  → $\lambda_1 \neq \lambda_2$




- $\lambda \in \mathbb{C} \setminus \mathbb{R}$
  ↪ choose a complex shift

- $\vec{x}_0$ has no comp. in $\vec{v}_1$ $\Leftrightarrow$ $\alpha_1 = 0$

  no problem: rounding error will make one

# What about Eigenvalues?

$Ax = -5x$ $\qquad \frac{\|Ax\|}{\|x\|}$

$Ax = (-5+3i)\vec{x}$

Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

$$A\vec{x} = \lambda\vec{x} \qquad (\vec{x} \neq \vec{0})$$

$$\frac{A\vec{x}}{\vec{x}} \text{ "}\approx\text{" } \begin{pmatrix} \lambda \\ \lambda \\ \lambda \\ \lambda \end{pmatrix}$$

$$\frac{x^T A x}{x^T x} = \lambda$$

Rayleigh quotient

**Demo:** Power Iteration and its Variants [cleared]

# Schur form: Motivation

For finding multiple eigenvalues, want factorization that allows access to all eigenvalues and eigenvectors.
Suggestions?