October 22, 2024

Announcements

- Exam 3
  - collage cheat sheet.
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Goals

QR it.

- Equivalence to orth. it.
- Cost

Krylov space methods

Review

- Schur form

$$A = Q U Q^T$$

- QR iteration

QR Iteration/QR Algorithm

orth. it

$X_0 = \cancel{\text{init guess}}\ A$

$Q_k R_k = X_k$

$X_{k+1} = A Q_k$

QR it

$\bar{X}_0 = A$

$\bar{Q}_k \bar{R}_k = \bar{X}_k \iff \bar{R}_k = \bar{Q}_k^H \bar{X}_k$

$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k$

$$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k = \bar{Q}_k^H \bar{X}_k \bar{Q}_k = \bar{Q}_k^H \cdots \bar{Q}_0^H A \bar{Q}_0 \cdots \bar{Q}_k \quad (*)$$

$$A^2 = \bar{Q}_0 (\bar{R}_0 \bar{Q}_0) \bar{R}_0 = \bar{Q}_0 \bar{Q}_1 \bar{R}_1 \bar{R}_0$$

Claim:

$$Q_k = \bar{Q}_0 \cdots \bar{Q}_k$$

$Q_k^H A Q_k = \tilde{X}_k = \bar{X}_{k+1} \rightsquigarrow$ toward

Proof sketch: Equivalence of QR iteration/Orth. iteration

Orthogonal Iteration (no bars)

- $X_0 := A$
  - $Q_0 R_0 := X_0$,
  - where we may choose $Q_0 = \bar{Q}_0$
  - $\hat{X}_0 = Q_0^H A Q_0 = Q_0^H Q_0 R_0 Q_0 = R_0 Q_0$
- $X_1 := AQ_0$
  - $Q_1 R_1 := X_1$, and because of * $X_1 = \underbrace{Q_0 Q_0^H}_{\mathrm{Id}} A Q_0 = Q_0 \bar{X}_1 =$ ** $Q_0 \bar{Q}_1 \bar{R}_1$ we may choose $Q_1 = Q_0 \bar{Q}_1 = \bar{Q}_0 \bar{Q}_1$.
- $\vdots$

Demo: QR Iteration [cleared]

QR Iteration (with bars)

- $\bar{X}_0 := A$
  - $\bar{Q}_0 \bar{R}_0 := A$ ✓
- $\bar{X}_1 := \bar{R}_0 \bar{Q}_0 = \hat{X}_0$
  - $\bar{Q}_1 \bar{R}_1 := \bar{X}_1$ **
- $\bar{X}_2 := \bar{R}_1 \bar{Q}_1$
  - $\bar{X}_2 = Q_1^H A Q_1 = \hat{X}_1$
- $\vdots$

QR Iteration: Forward *and* Inverse

$(A^{-1})^H = A^{-H} = (A^H)^{-1}$

QR iteration may be viewed as performing inverse iteration. How?

$\bar{X}_0 = A$

$\bar{Q}_k \bar{R}_k = \bar{X}_k$

$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k$

$\cdot^{-H}$

$\Rightarrow$

$\bar{X}_0^{-H} = A^{-H}$

$(\bar{Q}_k \bar{R}_k)^{-H} = \bar{X}_k^{-H}$

$\underbrace{\bar{Q}_k^{-H} \bar{R}_k^{-H}}_{QL \text{ fact.}} = (\bar{R}_k^{-1} \bar{Q}_k^{-1})^H =$

$\bar{X}_{k+1}^{-H} = \bar{R}_k^{-H} \bar{Q}_k^{-H}$

$x_{k+1} = A x_k$

$x_{k+1} = (A - \sigma I)^{-1} x_k$

QR Iteration: Incorporating a Shift

How can we accelerate convergence of QR iteration using shifts?

$$\bar{Q}_k \bar{R}_k = \bar{X}_k - \sigma_k I \iff \bar{R}_k = \bar{Q}_k^H X_k - \sigma_k \bar{Q}_k^H$$

$$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k + \sigma_k I$$

$$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k + \sigma_k I = \bar{Q}_k^H X_k \bar{Q}_k - \sigma_k \cancel{\bar{Q}_k^H} \bar{Q}_k + \sigma_k \cancel{I}$$

Demo: QR Iteration [cleared] (Shifted)

QR Iteration: Computational Expense

A full QR factorization at each iteration costs $O(n^3)$–can we make that cheaper?

```
Q, R = la.qr(X)
X = R @ Q

Qall = Qall @ Q
X
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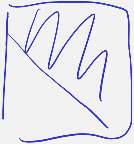
$H A H^T \rightarrow$ more zeros?

$H_2 H_1 A H_1^T H_2^T \cdots =$



upper Hessenberg

$O(n^3)$

Demo: Householder Similarity Transforms [cleared]

QR/Hessenberg: Overall procedure

$\bar{X}_0 = HAH^T$

$\bar{Q}_k \bar{R}_k = \bar{X}_k$

Givens QR $O(n^2)$

$\bar{X}_{k+1} = \underbrace{\bar{R}_k \bar{Q}_k}_{O(n^2)}$

Overall procedure:

1. Reduce matrix to Hessenberg form
2. Apply QR iteration using Givens QR to obtain Schur form

Why does QR iteration *stay* in Hessenberg form?

$\nabla \cdot UH = UH$

What does this process look like for symmetric matrices?

Krylov space methods: Intro

What subspaces can we use to look for eigenvectors?