October 29, 2024

Announcements

- Exam 3

Goals

Root Finding

Review

Solving Nonlinear Equations

#unk.

equations

What is the goal here?

$f: \mathbb{R}^n \to \mathbb{R}^n$

$g(\vec{x}) = h(\vec{x})$

LHS = RHS

$f(\vec{x}) = \vec{0}$

$\Rightarrow f(\vec{x}) = 0$

$= g(\vec{x}) - h(\vec{x})$

Showing Existence

How can we show existence of a root?

1D: $f:\mathbb{R}\to\mathbb{R}$ continuous

$\Rightarrow$ intermediate value thm. says $\exists$ root.

nD:

① $\vec{f}:\mathbb{R}^n\to\mathbb{R}^n$
$\Rightarrow f$ invertible at a point $\vec{x}_0$

"inverse function theorem" says f has an inverse in a ball around $\vec{x}_0$

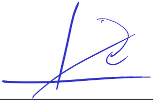
② $\vec{g}:\mathbb{R}^n\to\mathbb{R}^n$
fixed point: $\vec{g}(\vec{x}^*) = x^*$

"contraction mapping thm."
$\|g(\vec{x}) - g(\vec{y})\| \le \gamma \|x-y\|$
$0 \le \gamma < 1$

$\Rightarrow \exists$ fixed point: $\vec{g}(x^*) = x^*$

Sensitivity and Multiplicity

What is the sensitivity/conditioning of root finding? (1D)



solve $f(x)=0 \Leftrightarrow f^{-1}(0)$ f^{-1} has large abs. cond. $\Leftrightarrow f^{-1}$ has large derivative

f^{-1} ill-conditioned $\Leftrightarrow$ f^{-1} steep $\Leftrightarrow$ f flat $\Leftrightarrow$ f has a small deriv.

What are multiple roots?

$\frac{x\cdot f'(x)}{f(x)}$ $z=f^{-1}$ $\frac{x\cdot z'}{z}$ ← rel. cond

$f(x^*)=0$ $f'(x^*)=0$ ← double root

can't use because rel. perturbation of 0 is meaningless

How do multiple roots interact with conditioning?

bad!

Rates of Convergence

$\vec{x}_0, \vec{x}_1, \ldots \vec{x}_n,$

What is *linear convergence*? *quadratic convergence*?

$\vec{x}_n \to \vec{x}^*$ $\qquad \vec{e}_n := \vec{x}_n - x^*$

Examples: ① $\|\vec{e}_{k+1}\| \leq 0.9 \|e_k\|$ → guarantees conv.

② $\|\vec{e}_{k+1}\| \leq 3.4 \|e_k\|^2$

An it. method converges with rate r

$$\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C \begin{cases} > 0 \\ < \infty \end{cases}$$

$r=1$ → linear con.

$r=2$ → quadratic conv

$r>1$ → superlinear conv.

About Convergence Rates

Demo: Rates of Convergence [cleared]

Characterize linear, quadratic convergence in terms of the 'number of accurate digits'.

linear ⇒ gains const. # of digits
per it.

quadratic ⇒ doubles # of digits
every it.

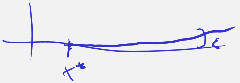
Stopping Criteria

Comment on the 'foolproof-ness' of these stopping criteria:

1. $|f(x)| < \varepsilon$ ('residual is small')
2. $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| < \varepsilon$
3. $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| \,/\, \|\boldsymbol{x}_k\| < \varepsilon$

Failure scenarios:

1



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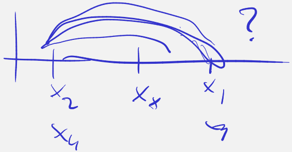
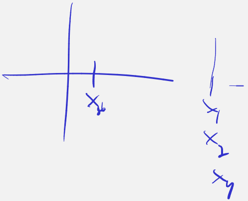
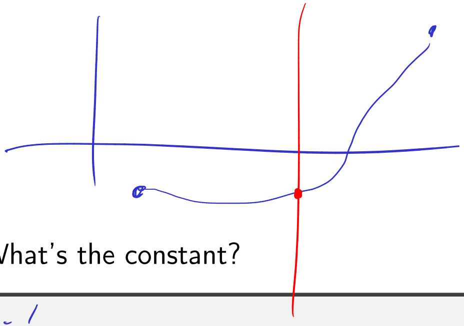




3 similar to 2

Bisection Method



Demo: Bisection Method [cleared]

What's the rate of convergence? What's the constant?

$r = 1$ $\quad C = \frac{1}{2}.$

Mini Review: Taylor's Theorem

"$h \to 0$"

$$f(x+h) = f(x) + f'(x)\cdot h + f''(x)\cdot\frac{h^2}{2} \ldots$$

Mini Review: Taylor's Theorem

$$\overset{g(x_k)}{f(\underset{=x^*}{x}+\underset{=e_k}{h})} = \overset{g(x^*)}{f(x)} + \overset{g'(x^*)}{f'(x)}h + \frac{f''(x)}{2!}h^2 + \cdots \quad \text{(requires } f \text{ analytic)}$$

Taylor with explicit remainder term ($\theta \in [x, x+h], f \in C^{k+1}$):

$$f(x+h) = f(x) + \cdots + \frac{f^{(k-1)}(x)}{(k-1)!}h^{k-1} + \frac{f^k(\theta)}{k!}h^k$$

Special case $k = 1$: Equivalent to mean value theorem:

$$f(x+h) = f(x) + f'(\theta)h \quad (\theta \in [x, x+h], f \in C^1)$$

With big-O truncation:

$$f(x+h) = f(x) + \cdots + \frac{f^{(k-1)}(x)}{(k-1)!}h^{k-1} + O(h^k) \quad (h \to 0, f \in C^{k+1})$$

Fixed Point Iteration

$$
\begin{aligned}
x_0 &= \langle\text{starting guess}\rangle \\
x_{k+1} &= g(x_k)
\end{aligned}
$$

(1D)

Demo: Fixed point iteration [cleared]

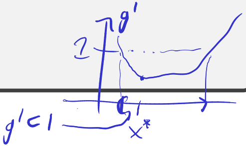
When does fixed point iteration converge? Assume g is smooth.

Suppose $g(x^*) = x^*$

$x^* + e_k$

$$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$$

$$= \underbrace{g'(\theta)}_{0 \le \quad < 1} e_k$$

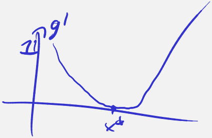


Fixed Point Iteration: Convergence cont'd.

Error in FPI: $e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$

If $g'(x^*) < 1$, then the iterative method will converge in a neighborhood of x^*.

What if $g'(x^*) = 0$



$$g(x_k) - g(x^*) = \cancel{\underbrace{g'(x^*)}_{=0} e_k} + g''(\theta) \cdot \frac{e_k^2}{2}$$

$\Rightarrow$ Iterative method with g is quadratically conv.

Newton's Method

Derive Newton's method.

Current guess: x_n

$$0 \approx f(x_n + h) \approx f(x_n) + f'(x_n) \cdot h + \dots$$

$$0 = f(x_n) + f'(x_n)\, h$$

$$\Leftrightarrow \quad h = -\frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n + h$$
$$= x_n - \frac{f(x_n)}{f'(x_n)}$$

Demo: Newton's method [cleared]

Convergence and Properties of Newton

What's the rate of convergence of Newton's method?

Drawbacks of Newton?

Demo: Convergence of Newton's Method [cleared]