Residual and Error

○ What is the residual vector of solving the linear system

\[ b = Ax \]

○ How do the (norms of the) residual vector \( r \) and the error \( \Delta x = x - \hat{x} \) relate to one another?

\[ \frac{\| \Delta x \|}{\| x \|} \leq \text{cond}(A) \cdot \frac{\| r \|}{\| A \| \| x \|} \]
Changing the Matrix

- So far, all our discussion was based on changing the right-hand side, i.e.

\[ Ax = b \quad \rightarrow \quad A\hat{x} = \hat{b}. \]

The matrix consists of FP numbers, too—it, too, is approximate. I.e.

\[ \hat{A} = A + \Delta A \]
\[ \hat{A} - A = \Delta A \]

What can we say about the error now?

\[ \Delta x = \hat{x} - x = A^{-1}(A\hat{x} - b) = A^{-1}(A\hat{x} - A\hat{x}) \]
\[ = -A^{-1}(\Delta A)\hat{x} \]

\[ \|\Delta x\| \leq \frac{\|A^{-1}\|\|\Delta A\|\|\hat{x}\|}{\|x\|} \]

\[ \leq \text{cond}(A) \frac{\|\Delta A\|}{\|A\|} \]
Changing Condition Numbers

- Once we have a matrix $A$ in a linear system $Ax = b$, are we stuck with its condition number? Or could we improve it?

\[
\text{With } M \text{ invertible: } \quad A\mathbf{x} = \mathbf{b} \quad | \quad M\quad \quad \quad \quad \quad x = My
\]

\[
MA\mathbf{x} = M\mathbf{b} \quad | \quad AMy = b
\]

- What is this called as a general concept?

Preconditioning

If $M \cdot D$ is diagonal, this is called diagonal preconditioning.
In-class activity: Matrix Conditioning II
2.2 Methods to Solve Systems
Solving Systems

- Want methods/algorithms to solve linear systems. Starting small, a kind of system that’s easy to solve has a ... matrix.
Triangular matrices

- Solve

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{22} & a_{23} & a_{24} \\
a_{33} & a_{34} \\
a_{44}
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{pmatrix}.
\]
Demo: Coding back-substitution
More General Matrices

- What about non-triangular matrices?
Gaussian Elimination

**Demo:** Vanilla Gaussian Elimination

- What do we get by doing Gaussian Elimination?

- How is that different from being upper triangular?

- What if we do not just eliminate downward but also upward?
Elimination Matrices

- What does this matrix do?

\[
\begin{pmatrix}
1 & 1 \\
\frac{-1}{2} & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{pmatrix}
\]