Fixed Point Iteration

\[ x_0 = \text{starting guess} \]
\[ x_{k+1} = g(x_k) \]

**Demo:** Fixed point iteration

- When does fixed point iteration converge? Assume \( g \) is smooth.

\[ x^* = g(x^*) \]

\[ e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*) \]
\[ \approx g'(x^*) (x_k - x^*) = g'(x^*) e_k \]

\[ 1/1 < 1 \]

If \( g'(x^*) = 0 \), Taylor gives

\[ e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*) \approx g''(x^*) (x_k - x^*)^2 / 2 \]
Newton’s Method

- Derive Newton’s method.

\[
\begin{align*}
\phi(x) &\approx 0 \quad \text{Can eval } f, f' \\
f(x_k + h) &\approx f(x_k) + hf'(x_k) = f(h) \\
0 &\approx f(h) = f(x_k) + hf'(x_k) \\
-f(x_k) &= hf'(x_k) \quad \Rightarrow \quad h = -\frac{f(x_k)}{f'(x_k)} \\
\end{align*}
\]

\[
\begin{align*}
x_{k+1} &= x_k + h = x_k - \frac{f(x_k)}{f'(x_k)} \\
\end{align*}
\]
\[ x_k = \frac{f(x_k)}{g(x_k)} \]

\[ g'(x) = \frac{f(x) f''(x)}{f'(x)^2} \]

**Suppose** \( x^* \) is \( f(x^*) = 0 \). **Quadratic conv.** if \( g'(x^*) = 0 \).

\[ g(x^*) = x^4 \]

**Disclaimer:** Assume \( f'(x_k) \neq 0 \). **For single roots:** quadratic convergence (at least locally!)

**For multiple roots:** linear conv.

**For single roots:** quadratic conv. (at least locally!)
Demo: Newton’s method
Demo: Convergence of Newton’s Method
Secant Method

- What would Newton without the use of the derivative look like?

\[ f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \]

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

\[ \frac{e_{k+1}}{e_k} \approx C \]
**Demo:** Secant Method

**Demo:** Convergence of the Secant Method

**In-class activity:** Nonlinear equations in 1D
‘Trusty’ Newton and Secant

- The linear approximations in Newton and Secant are only good locally. How could we use that?
5.3 Methods in $n$ Dimensions ("Systems of Equations")
Fixed Point Iteration

\[ x_0 = \text{starting guess} \]
\[ x_{k+1} = g(x_k) \]

- When does this converge?