6.2 Methods for unconstrained opt. in $n$ dimensions

\[ f : \mathbb{R}^n \rightarrow \mathbb{R} \]

\[ x_{k+1} = x_k + \alpha (-\nabla f(x_k)) \]

\[ \frac{\text{var}(x_k)}{} \]
Steepest Descent

- Given a scalar function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) at a point \( x \), which way is down?

1. \( x_0 \) (starting guess)
2. Define helper \( \psi(a) := f(x_0 + a(-\nabla f(x_0))) \)
3. Find min \( a' \) of \( \psi \)
4. \( x_{n+1} = x_n + a(-\nabla f(x_n)) \)
Demo: Steepest Descent
Newton’s method \((nD)\)

\[ \nabla f(x) \cdot \delta = \nabla f(x) \cdot \delta + \frac{1}{2} \delta^T H_f(x) \delta = 0 \]

\[ f(x + \delta) \approx f(x) + \nabla f(x) \cdot \delta + \frac{1}{2} \delta^T H_f(x) \delta = 0 \]

\[ \nabla^2 f(x) = \nabla f(x) + H_f(x) \]

\[ H_f(x) \delta = -\nabla f(x) \]

\[ \delta = -H_f(x)^{-1} \nabla f(x) \]

\[ s = -\frac{\partial^2 f(x)}{\partial x^2} \]

\[ x_0 = \text{starting guess} \]

\[ x_{n+1} = x_n - H_f(x_n)^{-1} \nabla f(x_n) \]
Demo: Newton’s method in $n$ dimensions

Demo: Nelder-Mead Method
6.3 Nonlinear Least Squares

\[ A \bar{x} \approx b \]

\[ \min \| A \bar{x} - b \|_2 \]

\[ f(\bar{x}) = y \]

\[ \min \| f(\bar{x}) - y \|_2 \]
Nonlinear Least Squares/Gauss-Newton

- What if the $f$ to be minimized is actually a 2-norm?

$$f(x) = \|r(x)\|_2^2, \quad r(x) = y - f(x)$$

**Demo:** Gauss-Newton

**In-class activity:** Optimization II

$$\psi(x) = \frac{1}{2} \|r(x)\|_2^2 = \frac{1}{2} r(x)^T r(x) = \sum_i r_i(x)^2$$

$$\frac{\partial}{\partial x_j} \psi = \frac{\partial}{\partial x_j} \left[ \sum_i r_i(x)^2 \right]$$

$$= \sum_i \left[ \frac{\partial}{\partial x_j} r_i(x) \right] r_i(x)$$

$$\nabla \psi = \sum_i r_i(x) \cdot \nabla r_i(x)$$
\[ H_\varphi(x) = \nabla x^T \nabla + \sum_i r_i H_{r_i} \]

\[ J^2 = J^2(x) \]

Small if residual close to 0
\[ \Rightarrow \text{ignore outright} \]

\[ x_0 = \text{starting guess} \]

\[ H_\varphi \delta = -\nabla \Phi \]

\[ x_{k+1} = x_k - H_\varphi^{-1} \nabla \Phi \]

\[ J^2 \delta s = -J^2 \nabla \Phi \]

\[ J^2 \nabla \Phi \nabla = -J^2 \nabla \]

\[ A^T A x = A^T b \]
\[ A x = b \]
6.4 Constrained Optimization
Constrained Optimization: Problem Setup

- Want \( x^* \) so that

\[
f(x^*) = \min_x f(x) \quad \text{subject to} \quad g(x) = 0
\]

No inequality constraints just yet. This is \textit{equality-constrained optimization}. Develop a necessary condition for a minimum.