8 Numerical Integration and Differentiation

8.1 Numerical Integration
Numerical Integration: About the Problem

- What is numerical integration? (Or ‘quadrature’?)

\[ \int_{a}^{b} f(x) \, dx \]

- What about existence and uniqueness?
  
  - Existence: \( f \) is integrable
    
    - Riemann
    - Lebesgue
  
  - Piecewise continuous
Conditioning

- Derive the (absolute) condition number for numerical integration.

\[
\hat{f}(x) = f(x) + e(x)
\]

\[
\left| \int f(x) \, dx - \int \hat{f}(x) \, dx \right| = \int e(x) \, dx \leq \int |e(x)| \, dx \\
\leq (b-a) \max_{x \in [a,b]} |e(x)|
\]

← absolute cond. number
8.1.1 Quadrature Methods
Interpolatory Quadrature

- Design a quadrature method based on interpolation.

\[ \int_a^b f(x) \, dx \approx \sum_{i=1}^{n} w_i f(x_i) \]

Interpolate at \((x_i)\):

\[ f(x) \approx \sum_{i=1}^{n} f(x_i) L_i(x) \]

Integrate:

\[ \int_a^b f(x) \, dx \approx \int_a^b \sum_{i=1}^{n} f(x_i) L_i(x) \, dx \]

Lagrange polynomial for \(x_i\)
- With monomials and equispaced pts, this is called Newton-Cotes quadrature.

- With Chebyshev nodes and Chebyshev polynomials, this is called Clenshaw-Curtis's quadrature.
Method of under coefficients

\[ b - a = \int_a^b 1 \, dx = \omega_1 \cdot 1 + \cdots + \omega_n \cdot 1 \]

\[ \frac{1}{2} (b^2 - a^2) = \int_a^b x \, dx = \omega_1 \cdot x_1 + \cdots + \omega_n \cdot x_n \]

\[ \frac{1}{k+1} (b^{k+1} - a^{k+1}) = \int_a^b x^k \, dx = \omega_1 \cdot x_1^k + \cdots + \omega_n \cdot x_n^k \]
\[ \frac{1}{2}(f(0) + f(1)) \]

\[ \frac{1}{4} f(0.5) \]
\[ \int_0^2 f(x) \, dx = \int_0^1 f(2\tau) \frac{dx}{dt} \, d\tau \]

\[ \tau = x/2 \]

\[ dx = 2 \, d\tau \]
Demo: Newton-Cotes weight finder
Examples and Exactness

- To what polynomial degree are the following rules exact?
  - **Midpoint rule**
    \[(b - a)f\left(\frac{a+b}{2}\right)\]
    
  - **Trapezoidal rule**
    \[\frac{b-a}{2}(f(a) + f(b))\]
    
  - **Simpson’s rule**
    \[\frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\]
8.1.2 Accuracy and Stability
Interpolatory Quadrature: Accuracy

- Let $p_{n-1}$ be an interpolant of $f$ at nodes $x_1, \ldots, x_n$ (of degree $n - 1$).

Recall

$$\sum_i \omega_i f(x_i) = \int_a^b p_n(x) \, dx.$$ 

What can you say about the accuracy of the method?

$$\|f\|_{\infty} = \max_{x \in [a,b]} |f(x)|$$

$$\left| \int f(x) \, dx - \sum_{i=1}^n \omega_i f(x_i) \right| = \left| \int f(x) \, dx - \int p_{n-1}(x) \, dx \right|$$

From interpolation:

$$\|f - p_{n-1}\|_{\infty} \leq C \cdot \|f^m\|_{L^\infty} h^n$$

$$\leq (b-a) \|f - p_{n-1}\|_{\infty}$$

$$\leq \left[ (b-a) C \|f^m\|_{L^\infty} h^n \right] = C \|f^m\|_{L^\infty} h^{n+1}$$
Demo: Accuracy of Simpson’s rule
Interpolatory Quadrature: Stability

- Let $p_n$ be an interpolant of $f$ at nodes $x_1, \ldots, x_n$ (of degree $n-1$)

Recall

$$\sum_i \omega_i f(x_i) = \int_a^b p_n(x)dx$$

What can you say about the stability of this method?