Examples and Exactness

- To what polynomial degree are the following rules exact?
  
  **Midpoint rule**
  \[(b - a)f\left(\frac{a+b}{2}\right)\]

  **Trapezoidal rule**
  \[\frac{b-a}{2}(f(a) + f(b))\]

  **Simpson’s rule**
  \[\frac{b-a}{6}\left( f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right)\]

  ![Diagrams showing exactness degrees for different rules]
8.1.2 Accuracy and Stability
**Interpolatory Quadrature: Accuracy**

- Let \( p_{n-1} \) be an interpolant of \( f \) at nodes \( x_1, \ldots, x_n \) (of degree \( n - 1 \)).

  Recall
  \[
  \sum_i \omega_i f(x_i) = \int_a^b p_{n-1}(x)dx.
  \]

  What can you say about the accuracy of the method?
Demo: Accuracy of Newton-Cotes
Interpolatory Quadrature: Stability

- Let $p_n$ be an interpolant of $f$ at nodes $x_1, \ldots, x_n$ (of degree $n - 1$).
  Recall
  \[
  \sum_i \omega_i f(x_i) = \int_a^b p_n(x) \, dx
  \]

  What can you say about the stability of this method?

Consider \( \hat{f}(x) = f(x) + e(x) \).

\[
\left| \sum_i \omega_i f(x_i) - \sum_i \omega_i \hat{f}(x_i) \right| = \left| \sum_i \omega_i e(x_i) \right|
\leq \|e\|_\infty \sum_i |\omega_i|
\]

\[\sum \omega_i = b-a\]
About Newton-Cotes

- What’s not to like about Newton-Cotes quadrature?
  - Stability for high point counts: \( \times \)
  - \( V_{dm} \) is ill-conditioned: \( \times \)
  - Inherits all issues from high-order poly interp.
  - Hard to extend to high # points

Ideas: Lots of tiny sub-quadratures \( \text{“Composite quad.”} \)

Make high-order work

\( \text{Gaussian quadrature} \)
8.1.3 Composite Quadrature

- High-order polynomial interpolation requires a high degree of smoothness of the function.

**Idea:** Stitch together multiple lower-order quadrature rules to alleviate smoothness requirement.

e.g. trapezoidal

\[ \int_a^b f(x) \, dx \approx \sum_{j=1}^{n} \omega_{j,i} f(x_{i,j}) \]

What can we say about the error in this case?

**Single interval:** \[ | \int_a^b (f - p_{n-1}) | \leq C \cdot h^{n+1} \| f^{(n)} \|_{\infty} \]

**Multi-interval:** \[ \left| \int_a^b f(x) \, dx - \sum_{j=1}^{n} \sum_{i=1}^{m} \omega_{j,i} f(x_{i,j}) \right| \leq C \cdot \| f^{(n)} \|_{\infty} \sum_{j=1}^{n} (a_j - a_{j-1})^{n+1} \]

\[ \leq C \cdot \| f^{(n)} \|_{\infty} \sum_{j=1}^{n} h^n (a_j - a_{j-1}) \]

\[ = C \cdot \| f^{(n)} \|_{\infty} h^n \left( \sum_{j=1}^{n} (a_j - a_{j-1}) \right) = \frac{b-a}{\sum_{j=1}^{n} (a_j - a_{j-1})} \]
8.1.4 Gaussian Quadrature

- So far: nodes chosen from outside.
  Can we gain something if we let the quadrature rule choose the weights, too? **Hope:** More design freedom $\implies$ Exact to higher degree.

**Demo:** Gaussian quadrature weight finder

$$\int_{a}^{b} x^k \, dx = \alpha_1 x^k + \cdots + \alpha_n x^k$$
8.2 Numerical Differentiation