Initial Value Problems: Problem Statement

- Want: Function $y: [0, T] \rightarrow \mathbb{R}^n$ so that
  - $y^{(k)}(t) = f(t, y, y', y'', \ldots, y^{(k-1)})$ (explicit)
  - or
  - $f(t, y, y', y'', \ldots, y^{(k)}) = 0$ (implicit)

are called explicit/implicit $k$th-order ordinary differential equations (ODEs). Give a simple example.

- Not uniquely solvable on its own. What else is needed?
  - Need IC.

\[ y'(t) = \alpha y(t) \]
\[ y(0) = y_0 \]
\[ y(t) = y_0 e^{\alpha t} \]
Reducing ODEs to First-Order Form

- A $k$th order ODE can always be reduced to first order. Do this in this example:

$$y''(t) = f(y)$$

By introducing enough extra variables, we can convert order to 1.
Properties of ODEs

- What is an autonomous ODE?
- What is a linear ODE?
- What is a linear and homogeneous ODE?
- What is a constant-coefficient ODE?

\[ \dot{y}(t) = f(t, y(t)) \]

\[ \dot{y}(t) = A(t) \dot{y}(t) + b \]

\[ \dot{y}(t) = A(t) y(t) \]

\[ \begin{cases} y'(u) = 3 + y \\ \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right)'(t) = \left( \begin{array}{c} 3 y_2 \\ y_1 \end{array} \right) \end{cases} \]
9.1 Existence, Uniqueness, Conditioning
Existence and Uniqueness

Consider the perturbed problem

\[
\begin{align*}
  y'(t) &= f(y) \\
y(t_0) &= y_0
\end{align*}
\begin{align*}
  \hat{y}'(t) &= f(\hat{y}) \\
  \hat{y}(t_0) &= \hat{y}_0
\end{align*}
\]

Then if \( f \) is **Lipschitz continuous** (has ‘bounded slope’), i.e.

\[
\| f(y) - f(\hat{y}) \| \leq L \| y - \hat{y} \|
\]

(where \( L \) is called the **Lipschitz constant**), then...

○ What does this mean for uniqueness?

There exists a solution in a neighborhood of \( t_0 \)

\[
\| y(t) - \hat{y}(t) \| \leq C \cdot e^{L(t-t_0)} \| y_0 - \hat{y}_0 \|
\]

\( y'(t) = f(y) \)
Conditioning

Unfortunate terminology accident: “Stability” in ODE-speak
To adapt to conventional terminology, we will use ‘Stability’ for

- the conditioning of the IVP, and
- the stability of the methods we cook up.

Some terminology:

- An ODE is **stable** if and only if...
  
  \[
  \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \| y(0) - \hat{y}(0) \| < \delta \Rightarrow \| \hat{y}(t) - y(t) \| < \varepsilon \text{ for all } t \geq t_0
  \]

- An ODE is **asymptotically stable** if and only if
  
  \[
  \| \hat{y}(t) - y(t) \| \to 0 \text{ as } t \to \infty
  \]
Example 1: Scalar, Constant-Coefficient

\[
\begin{cases}
y'(t) = \lambda y \\
y(0) = y_0
\end{cases}
\quad \text{where} \quad \lambda = a + ib
\]

○ Solution?

\[y(t) = y_0 e^{\lambda t}\]

○ When is this stable?

If \(\text{Re} \lambda \leq 0\):

\[|y(t)| = |y_0 e^{at} e^{ibt}| = |y_0| e^{\text{Re}(e^{ibt})} = |y_0| e^{a|\text{Im}(\lambda)|} \]

\(a < 0 \quad \checkmark \)

\(a > 0 \quad \xmark \)

\(a = 0 \quad \checkmark \)
\[ UA V^{-1} = 0 \quad B = V^{-1} \sim V = B^T \sim B^T A B = 0 \]

**Example II: Constant-Coefficient System**

\[
\begin{cases}
    y'(t) = Ay(t) \\
    y(t) = y_0
\end{cases}
\]

Assume \( V^M AV^{-1} = D = \text{diag}(\lambda_1, \ldots, \lambda_n) \) diagonal.

- How do we find a solution?
- When is this stable?

Define \( \mathbf{\tilde{w}}(t) = V \mathbf{y}(t) \)

\[
\mathbf{\tilde{w}}'(t) = V \mathbf{y}'(t) = V A \mathbf{y}(t) = V A V^{-1} \mathbf{\overline{w}}(t) = 0
\]

\[
\mathbf{\overline{w}}(t) = \lambda_1 \mathbf{w}(t)
\]

\[\text{if } \Re \lambda_i \leq 0.\]

(because we can look at stability of \( \mathbf{\overline{w}} \).)
9.2 Numerical Methods (I)

\[ x^2 + 3x - 5 = 0 \]
\[ y^2 + 3y - 5 = 0 \]
Euler’s Method

- Discretize the IVP

\[
\begin{aligned}
\begin{cases}
y'(t) &= f(y) \\
y(t) &= y_0
\end{cases}
\end{aligned}
\]

- Discrete times: \( t_1, t_2, \ldots, \) with \( t_{i+1} = t_i + h \)
- Discrete function values: \( y_k \approx y(t_k) \).
Demo: Forward Euler stability
9.3 Accuracy and Stability