$\|A\|$

$$\|Ax\| \le \underbrace{\|A\|}\,\|x\|$$

## Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

1. $\|A\| > 0 \Leftrightarrow A \neq \mathbf{0}$. ✓
2. $\|\gamma A\| = |\gamma| \|A\|$ for all scalars $\gamma$. ✓
3. Obeys triangle inequality $\|A + B\| \leqslant \|A\| + \|B\|$ ✓

- But also some more properties that stem from our definition:

$$\|Ax\| \leq \|A\| \, \|x\|$$

$$\|AB\| \leq \|A\| \, \|B\|$$

## Conditioning

- Now, let's study conditioning of solving a linear system

$$\underset{\text{out}}{A\boldsymbol{x}} = \overset{\text{in}}{\boldsymbol{b}}.$$

$\otimes$ $\|Ax\| \le \|A\|\|x\|$

$A(x+\Delta x) = (b+\Delta b)$

$A\Delta x = \Delta b$

$\Delta x = A^{-1}\Delta b$

cond($Ax=b$)

$$\frac{\text{rel. err. in output}}{\text{rel. err. in input}} = \frac{\|\Delta x\|/\|x\|}{\|\Delta b\|/\|b\|} = \frac{\|\Delta x\|\,\|b\|}{\|\Delta b\|\,\|x\|}$$

$$= \frac{\|A^{-1}\Delta b\|\,\|Ax\|}{\|\Delta b\|\,\|x\|} \overset{\otimes}{\le} \|A^{-1}\|\cdot\|A\|\cdot\frac{\cancel{\|\Delta b\|}\,\cancel{\|x\|}}{\cancel{\|\Delta b\|}\,\cancel{\|x\|}}$$

$\|A^{-1}\Delta b\| \le \|A^{-1}\|\,\|\Delta b\|$

cond$(A) = \|A\|\cdot\|A^{-1}\|$

$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$

rel. to a specific norm

cond$_2(A) = \|A\|_2\|A^{-1}\|_2$

for non-invertible $A$:

cond$(A) = \infty$

**Demo:** Condition number visualized
**In-class activity:** Conditioning
**Demo:** Conditioning of $2 \times 2$ Matrices

$$1 = \|I\| = \|A A^{-1}\| \leq \|A\| \, \|A^{-1}\| = \operatorname{cond}(A)$$

$$1 \leq \operatorname{cond}(A)$$

$\mathrm{cond}(A)$

$\mathrm{cond}(A^{-1}) = \|(A^{-1})^{-1}\| \, \|A^{-1}\| = \|A\| \cdot \|A^{-1}\|$

$Ax = b \qquad\qquad A^{-1} b = x$

---

$1 = \|I\| = \|AA^{-1}\| \leq \|A\| \, \|A^{-1}\|$

---

$\|A\| = \left\| \begin{pmatrix} 100 & & \\ & 13 & \\ & & 0.5 \end{pmatrix} \right\|_2 = 100 \qquad \|A^{-1}\| = \left\| \begin{pmatrix} 0.01 & & \\ & 1/13 & \\ & & 2 \end{pmatrix} \right\| = 2$

$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{vmatrix} 100x \\ 13y \\ 0.5z \end{vmatrix}$

## Residual and Error

- What is the residual vector of solving the linear system

$$\boldsymbol{b} = A\boldsymbol{x}?$$

$$r = b - A\hat{x}$$

- How do the (norms of the) residual vector $\boldsymbol{r}$ and the error $\Delta \boldsymbol{x} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ relate to one another?

$$\|\Delta x\| = \| x - \hat{x} \|$$

$$= \| A^{-1}(b - A\hat{x}) \|$$

$$= \| A^{-1} r \|$$

div. by $\|\hat{x}\|$

$$\frac{\|\Delta x\|}{\|\hat{x}\|} = \frac{\|A^{-1}r\|}{\|\hat{x}\|} \leq \frac{\|A^{-1}\| \, \|r\|}{\|\hat{x}\|} = \frac{\|A\|}{\|A\|} \|A^{-1}\|$$

$$\operatorname{cond}(A) \frac{\|r\|}{\|A\| \, \|\hat{x}\|}$$

## Changing the Matrix

- So far, all our discussion was based on changing the right-hand side, i.e.

$$A\boldsymbol{x} = \boldsymbol{b} \quad \rightarrow \quad A\hat{\boldsymbol{x}} = \hat{\boldsymbol{b}}.$$

The matrix consists of FP numbers, too–it, too, is approximate. I.e.

$$A\boldsymbol{x} = \boldsymbol{b} \quad \rightarrow \quad \hat{A}\hat{\boldsymbol{x}} = \boldsymbol{b}.$$

What can we say about the error now?

## Changing Condition Numbers

- Once we have a matrix $A$ in a linear system $A\boldsymbol{x} = \boldsymbol{b}$, are we stuck with its condition number? Or could we improve it?

- What is this called as a general concept?