$\mathrm{cond}(A) = \|A\| \|A^{-1}\|$

Residual and Error

- What is the residual vector of solving the linear system

$$\boldsymbol{b} = A\boldsymbol{x}?$$

- How do the (norms of the) residual vector $\boldsymbol{r}$ and the error $\Delta\boldsymbol{x} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ relate to one another?

$$\frac{\|\Delta x\|}{\|x\|} \leq \mathrm{cond}(A) \cdot \frac{\|r\|}{\|A\| \|\hat{x}\|}$$

Changing the Matrix

- So far, all our discussion was based on changing the right-hand side, i.e.

$$A\boldsymbol{x} = \boldsymbol{b} \quad \to \quad A\hat{\boldsymbol{x}} = \hat{\boldsymbol{b}}.$$

The matrix consists of FP numbers, too–it, too, is approximate. I.e.

$$A\boldsymbol{x} = \boldsymbol{b} \quad \to \quad \hat{A}\hat{\boldsymbol{x}} = \boldsymbol{b}.$$

$$\hat{A} = A + \Delta A$$

$$\hat{A} - A = \Delta A$$

What can we say about the error now?

$$\Delta x = \hat{x} - x = A^{-1}(A\hat{x} - b) = A^{-1}(A\hat{x} - \hat{A}\hat{x})$$

$$(A - \hat{A})\hat{x}, \quad A - \hat{A} = -\Delta A$$

$$= -A^{-1}(\Delta A)\hat{x}$$

$$\frac{\|\Delta x\|}{\|\hat{x}\|} \le \|A^{-1}\| \, \frac{\|A\|}{\|A\|} \, \|\Delta A\|$$

$$= \operatorname{cond}(A) \cdot \frac{\|\Delta A\|}{\|A\|}$$

Changing Condition Numbers

- Once we have a matrix A in a linear system $A\boldsymbol{x} = \boldsymbol{b}$, are we stuck with its condition number? Or could we improve it?

With M invertible: $Ax = b \quad | \, M \cdot$ | $x = My$

$MAx = Mb$ | $AMy = b$

- What is this called as a general concept?

Preconditioning

"Left" "Right"

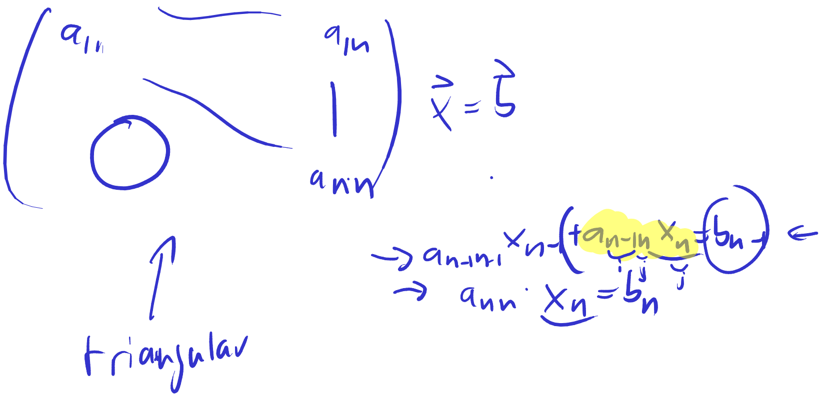
If $M = D$ is diagonal, this is called diagonal preconditioning.

In-class activity: Matrix Conditioning II

2.2 Methods to Solve Systems

Solving Systems

- Want methods/algorithms to solve linear systems. Starting small, a kind of system that's easy to solve has a ... matrix.



Triangular matrices

- Solve

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

Demo: Coding back-substitution

More General Matrices

- What about non-triangular matrices?

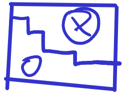
GE

Gaussian Elimination

Demo: Vanilla Gaussian Elimination

- What do we get by doing Gaussian Elimination?

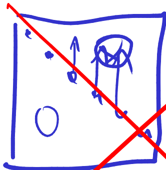
REF



- How is that different from being upper triangular?

REF could have zeros on the diag.

- What if we do not just eliminate downward but also upward?



computationally inefficient

Gauss-Jordan elimination

Elimination Matrices

- What does this matrix do?

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ -\frac{1}{2} & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$