LU: Failure Cases?

- Is LU/Gaussian Elimination bulletproof?

$$\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

- What can be done to get something *like* an LU factorization?

Fixing nonexistence of LU

- How do we capture 'row switches' in a factorization?

- What does this process look like then? $\begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} A & A & A & A \\ B & B & B & B \\ C & C & C & C \\ D & D & D & D \end{pmatrix}$

"Vanilla" LU: $M_\ell \cdots M_3 M_2 M_1 A$

w/ Partial Pivoting:

$$M_3 P_3 M_2 P_2 M_1 P_1 A = U$$

$$A = \underbrace{P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} P_3^{-1} M_3^{-1}}_{L?} U$$

Demo: LU with Pivoting (Part I)

What about the L in LU?

- Sort out what LU with pivoting looks like.

$$\underbrace{P}_{\smile}A = LU \checkmark$$

$$M_3 P_3 M_2 P_2 M_1 P_1 A = U$$

$$L_3 = M_3$$

$$L_2 = P_3 M_2 P_3^{-1}$$

$$L_1 = P_3 P_2 M_1 P_2^{-1} P_3^{-1}$$

$$L_3 L_2 L_1 P_3 P_2 P_1$$

$$= M_3 P_3 M_2 \cancel{P_3^{-1}} \cancel{P_3} P_2 M_1 \cancel{P_2^{-1}} \cancel{P_3^{-1}} \cancel{P_3} \cancel{P_2} P_1$$

$$= M_3 P_3 M_2 P_2 M_1 P_1$$

$$L_3 L_2 L_1 P_3 P_2 P_1 A = U$$

$$\underbrace{P_3 P_2 P_1}_{P} A = \underbrace{L_1^{-1} L_2^{-1} L_3^{-1}}_{L} U$$

Demo: LU with partial pivoting (Part II)

Computational Cost

- What is the computational cost of multiplying two $n \times n$ matrices?

each dot prod: $O(n)$ ops.

n^2 of them

$O(n^3)$

- What is the computational cost of carrying out LU factorization on an $n \times n$ matrix?

$$M_3 P_3 M_2 P_2 M_1 P_1 A = U$$

n matrix-matrix products: $\cancel{O(n^4)}$

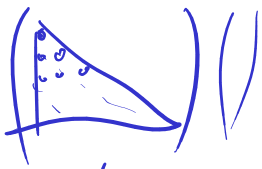
$n \times$ $M(\ldots) \to O(n^2)$

$P(\ldots) \to O(n) \quad \to \quad O(n^3) \to (2n)^3 = 2^3 \cdot n^3 = 8n$

Demo: Complexity of Mat-Mat multiplication and LU

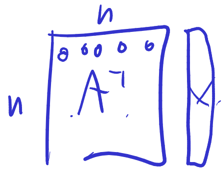
In-class activity: Pivoting and Cost

Cost of fw/bw subst:



$O(n^2)$

$Ax=b$



n^2

$$(A^{-1}x)_i = \sum_{j=1}^{n} (A^{-1})_{ij} x_j$$

More cost concerns

- What's the cost of solving $A\boldsymbol{x} = \boldsymbol{b}$?

- What's the cost of solving $A\boldsymbol{x} = \boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n$?

- What's the cost of finding A^{-1}?

Cost: Worrying about the Constant, BLAS

$O(n^3)$ really means

$$\alpha \cdot n^3 + \beta \cdot n^2 + \gamma \cdot n + \delta.$$

All the non-leading and constants terms swept under the rug. But: at least the leading constant ultimately matters.