Pseudoinverse

$Ax \cong b$

- What is the pseudoinverse of A?

- What can we say about the condition number in the case of a tall-and-skinny, full-rank matrix?

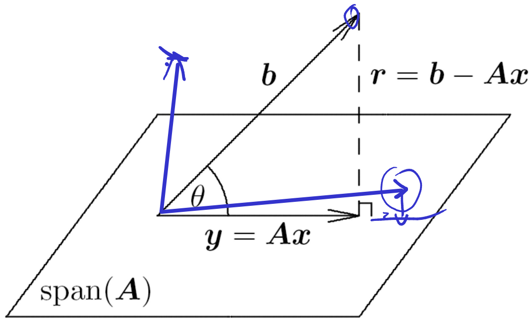
- What does all this have to do with solving least squares problems?

3.2 Sensitivity and Conditioning

Sensitivity and Conditioning of Least Squares

$(A^TA)^{-1}A^T$

$\frac{\|\Delta x\|}{\|x\|} \le \text{cond} \frac{\|\Delta b\|}{\|b\|}$



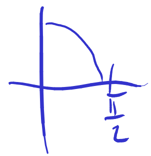
- How sensitive is it?

$\cos\theta = \frac{\|Ax\|}{\|b\|}$

$\frac{\|\Delta x\|}{\|x\|} \le \text{cond}(A) \frac{1}{\cos\theta} \frac{\|\Delta b\|}{\|b\|}$

- What values of θ are bad?

$\theta \approx \frac{\pi}{2} \rightsquigarrow \cos\theta$



- Any comments regarding dependencies?

Unlike for lin. sys., this bound depends on A <u>and</u> b.

- What about changes in the matrix?

$$\frac{\|\Delta x\|}{\|x\|} \le \left[\operatorname{cond}(A)^2 \tan(\theta) + \operatorname{cond}(A) \right] \cdot \frac{\|\Delta A\|}{\|A\|}$$

3.3 Solving Least Squares

Ideas for Solving Least Squares

- Tell me about the augmented system method.

Solve for x, r.

$$\begin{aligned} r + Ax &= b \\ A^T r &= 0 \end{aligned} \rightsquigarrow \begin{pmatrix} \mathrm{Id} & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

Least-squares by Transformation

- Want a matrix Q so that

$$QAx \cong Qb$$

has the same solution as

$$Ax \cong b. \rightsquigarrow \|Ax-b\|_2$$

I.e. want

$$\|Q(Ax-b)\|_2 = \|Ax-b\|_2.$$

What type of matrix does that? Any invertible one?

Orth. matrices: $Q^TQ=I$ and $QQ^T=I$

$$\|Qv\|_2^2 = (Qv)^T(Qv) = v^T\underbrace{Q^TQ}_{I}v = v^Tv = \|v\|_2^2$$

$A=QR$

Orthogonal Matrices

- What's an orthogonal (=orthonormal) matrix?

- Are orthogonal projectors orthogonal?

- Now what about that norm property?

Simpler Problems: Triangular

- Would we win anything from transforming a least-squares system to upper triangular form?

$m > n$

$$m\ \underbrace{\begin{pmatrix} | & | & | \end{pmatrix}}_{QR}^{\,n} \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} b \end{pmatrix} \rightsquigarrow \left\| \begin{pmatrix} R \\ 0 \end{pmatrix} \begin{pmatrix} x \end{pmatrix} - \begin{pmatrix} Q^T b \end{pmatrix} \begin{matrix} \text{top} \\ \text{bottom} \end{matrix} \right\|_2 \to \min$$

- If so, how would we minimize the residual norm?

$$A x \cong b$$

$$Q^T A x \cong b$$

$$\underbrace{Q^T Q}_{I} R x \cong b$$

$$R \tilde{x} \cong b$$

$$\|Rx - Q^T b\|_2^2 = \|[Rx - Q^T b]_{\text{top}}\|_2^2 + \|[Rx - Q^T b]_{\text{bottom}}\|_2^2$$

$$= \|Rx - (Q^T b)_{\text{top}}\|_2^2 \leftarrow 0 \quad x = R^{-1} b$$

$$+ \| - (Q^T b)_{\text{bottom}}\|_2^2$$

Computing QR

- Gram-Schmidt
- Householder Reflectors
- Givens Rotations

Latter two similar to LU:

- Successively zero out below-diagonal part
- But: using orthogonal matrices

Demo: Gram-Schmidt–The Movie
Demo: Gram-Schmidt and Modified Gram-Schmidt
Demo: Keeping Track of Coefficients in Gram-Schmidt
In-class activity: QR

Householder Transformations

- Find an *orthogonal* matrix Q that zeroes out the lower part of a column vector $\boldsymbol{a}$.