Givens Rotations

- If reflections work, can we make rotations work, too?

$A^T = A$

$A^TA = I$

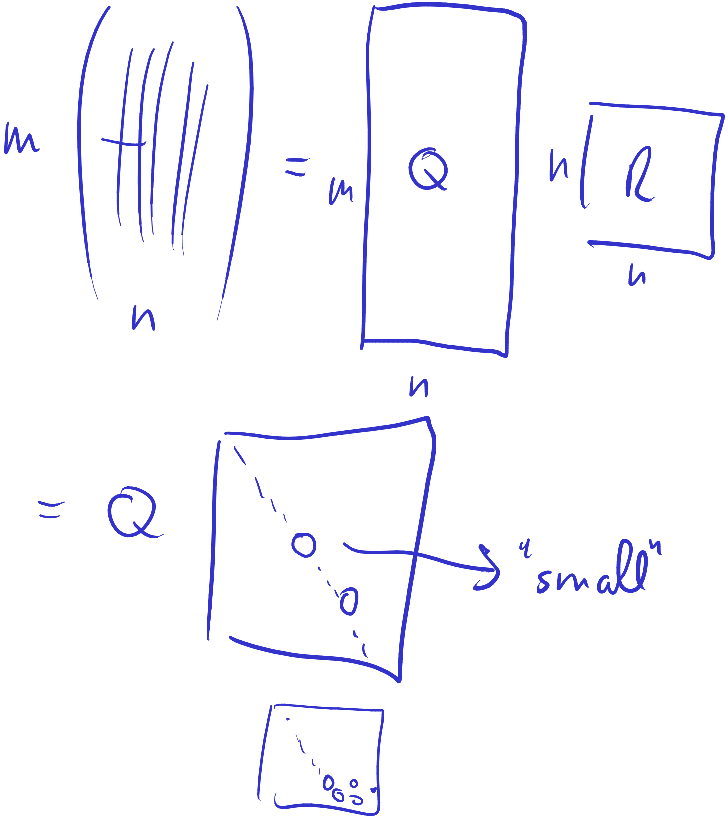
$$\underbrace{\begin{pmatrix} c & s \\ -s & c \end{pmatrix}}_{\text{orthogonal}} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \pm\sqrt{a_1^2 + a_2^2} \\ 0 \end{pmatrix}$$

Householder: zeros whole col.

Givens rot: zeros one entry.

Rank-Deficient Matrices and QR

- What happens with QR for rank-deficient matrices?



$PA = LU$

$AP = QR$

$= Q \begin{pmatrix} R \\ 0 \end{pmatrix}$

"Rank-Revealing QR" RRQR.

Rank-Deficient Matrices and Least-Squares

- What happens with Least Squares for rank-deficient matrices?

$$A\boldsymbol{x} \cong \boldsymbol{b}$$

  - QR still finds a solution with minimal residual
  - By QR it's easy to see that least squares with a short-and-fat matrix is equivalent to a rank-deficient one.
  - **But:** No longer unique. $\boldsymbol{x}+\boldsymbol{n}$ for $\boldsymbol{n} \in N(A)$ has the same residual.
  - **In other words:** Have more freedom
  Or: Can demand another condition, for example:
    - Minimize $\|\boldsymbol{b} - A\boldsymbol{x}\|_2^2$, *and*
    - minimize $\|\boldsymbol{x}\|_2^2$, simultaneously.

  Unfortunately, QR does not help much with that $\rightarrow$ Need better tool.

What happens to least sqaures (with QR) if A does not have full rank?

$$Ax \cong b$$

$$QRx \cong b$$

$$Rx \cong Q^T b$$

$$A\vec{n} = \vec{0} \qquad \vec{n} \in N(A)$$

$$\| A\vec{x} - \vec{b} \|_2 \to \min$$

$$= \| A(\vec{x} + \vec{n}) - b \|_2$$

Solution is no longer uniquely determined.
-> Have an additional freedom
-> Can ask for another condition

$$\|x\|_2 \to \min \quad \text{and} \quad \| A\vec{x} - \vec{b} \|_2 \to \min$$

$$Av_1 = U\Sigma V^T v_1 = U\Sigma \begin{pmatrix} \sigma_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \sigma_1 u_1 \qquad \|Av_1\| = \sigma_1$$

Singular Value Decomposition (SVD)

- What is the Singular Value Decomposition of an $m \times n$ matrix?

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T \qquad A = U \Sigma V^T \qquad A: m \times n$$

$$\begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ & & & \end{pmatrix}$$

left singular v. $\rightarrow$ $U: m \times m \leftarrow$ orthogonal

singular values $\rightarrow$ $\Sigma: m \times n \leftarrow$ diagonal

right sing. v. $\rightarrow$ $V: n \times n \leftarrow$ orthogonal

$$|\sigma_1| \geq |\sigma_2| \geq \dots |\sigma_n| \geq 0$$

$$U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{pmatrix} \qquad V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

SVD: What's this thing good for?

$A = U\Sigma V^T$

-

$\|A\|_2 = \sigma_1$

$A^{-1} = (U\Sigma V^T)^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$

$\|A^{-1}\|_2 = 1/\sigma_n$

$\operatorname{cond}_2(A) = \sigma_1/\sigma_n$

$\operatorname{rank}(A) =$ # nonzero sing. values

$\operatorname{num\,rank}_\varepsilon(A) =$ # sing. values $> \varepsilon$

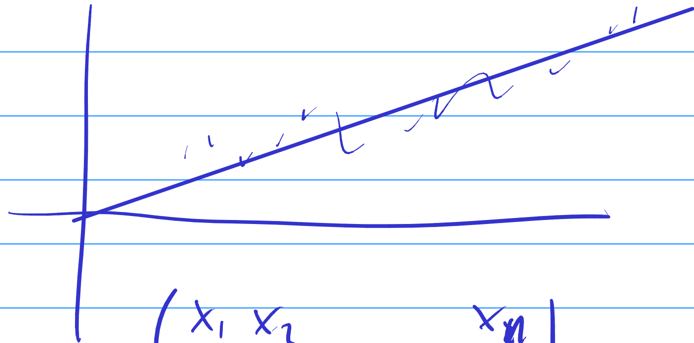
Low-rank approximation

Eckart-Young Theorem

$k < r = \text{rank}(A)$. If

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

then $\min_{\text{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2$



$$\begin{pmatrix} x_1 & x_2 & & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

In-class activity: SVDs

Comparing the Methods

Multiplications to solve least squares with A an $m \times n$ matrix:

- Form: A^TA: $n^2m/2$
  Solve with A^TA: $n^3/6$
- Solve with Householder: $m\,n^2 - n^3/3$
- If $m \approx n$, about the same
- If $m \gg n$: Householder QR requires about twice as much work as normal equations
- SVD: $m\,n^2 + n^3$ (with a large constant)