4 Eigenvalue Problems

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

- $\boldsymbol{x} \neq \mathbf{0}$ is called an eigenvector of A if there exists a λ so that

$$A\boldsymbol{x} = \lambda \boldsymbol{x}.$$

- In that case, λ is called an eigenvalue.
- The set of all eigenvalues $\lambda(A)$ is called the spectrum.
- The spectral radius is the magnitude of the biggest eigenvalue:

$$\rho(A) = \max\{|\lambda| : \lambda(A)\}$$

Finding Eigenvalues

- How do you find eigenvalues?

$A : n \times n$

$Ax = \lambda x$

$\Leftrightarrow (A - \lambda I)x = 0$

$\Leftrightarrow A - \lambda I$ is singular

$\Leftrightarrow \det(A - \lambda I) = 0$ $\Leftarrow$ polynomial of deg. n

deg $n \geq 5$ $\rightarrow$ no formula for the roots

Multiplicity

- What is the multiplicity of an eigenvalue?

$\det(A-\lambda I) = (\lambda-\lambda_1)^{n_1} \cdots (\lambda-\lambda_k)^{n_k}$ ← Algebraic multiplicities

# of lin indep. eigenvec. for λ_i ← Geometric multiplicity

AM $\geq$ GM

An Example

- Give characteristic polynomial, eigenvalues, eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}. \qquad \begin{pmatrix} \lambda-1 & 1 \\ & \lambda-1 \end{pmatrix}$$

CP: $(1-\lambda)^2$ Eigval 1 w/ alg. multipl. 2

$$\begin{pmatrix} x+y \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{matrix} x+y=x \rightsquigarrow y=0 \\ \rightsquigarrow y=y \end{matrix}$$

$\rightsquigarrow$ geom. multiplicity 1

Diagonalizability

- When is a matrix called diagonalizable?

$$X = \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{pmatrix} \quad \text{eigenvectors}$$

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$AX = XD$$

$$\rightsquigarrow \quad X^{-1}AX = D \quad \rightsquigarrow \quad A = XDX^{-1}$$

Definition: Two matrices C, E are called similar if there exists a T: $C = T^{-1}ET$ ← similarity transform

A matrix A is diagonalizable if it is similar to a diagonal matrix.

Similarity transforms leave eigenvalues unchanged:

$$A = T^{-1} B T$$

$$Ax = \lambda x \quad (x \neq 0)$$

$$T^{-1} B T x = \lambda x$$

$$B \underbrace{Tx}_{y} = \lambda \underbrace{Tx}_{y} \rightarrow By = \lambda y$$

4.1 Sensitivity

Sensitivity

- Assume A not defective. Suppose $X^{-1}AX = D$. Perturb

$$A \to A + E$$

What happens to the eigenvalues?

not necessarily diagonal

$$X^{-1}(A+E)X = D + F$$

$$(D+F)\vec{v} = \mu\vec{v}$$

$$F\vec{v} = (\mu I - D)\vec{v}$$

$$(\mu I - D)^{-1}F\vec{v} = \vec{v}$$

assume: μ is not one of the original, "true" eigenvalues

$$\|v\| = \|(\mu I - D)^{-1}F\vec{v}\|$$

$$\|v\| \le \|(\mu I - D)^{-1}\| \, \|F\| \, \|\vec{v}\|$$

$$\frac{\|v\|}{\|(\mu I - D)^{-1}\|} \le \|F\| \, \|v\|$$

$$\frac{\|v\|}{\|(\mu I - D)^{-1}\|} \le \|F\|\,\|v\| \qquad X^{-1}EX = F$$

$$= \|X^{-1}EX\|\,\|v\|$$

$$\frac{\cancel{\|v\|}}{\|(\mu I - D)^{-1}\|} \le \|X^{-1}\|\,\|X\|\,\|E\|\,\cancel{\|v\|}$$

$$\|(\mu I - D)^{-1}\|^{-1} \le \underline{\text{cond}(X)\,\|E\|}$$

distance between mu and the closest eigenvalue to mu

$$\begin{pmatrix} (\mu-\lambda_1)^{-1} & & \\ & \ddots & \\ & & (\mu-\lambda_n)^{-1} \end{pmatrix}$$

$$\|(\mu I - D)^{-1}\|$$

gives us the 1/(difference between mu and the closest eigenvalue to mu)

$$Ax = \lambda x$$

$$A(2x) = \lambda(2x)$$

Q orthogonal

$$\|Qx\|_2 = \|x\|_2$$

$$\|Qx\|_2 \le \|Q\| \, \|x\|_2$$

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random $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$

$$\left.\begin{array}{l} Ax_1 = \lambda_1 x_1 \\ \vdots \\ Ax_n = \lambda_n x_n \end{array}\right) \quad \begin{array}{c} |\lambda_1| \\ \geq \\ \vdots \\ \geq \\ |\lambda_n| \geq 0 \end{array}$$

$$A^{1000} x / \lambda_1^{1000}$$

$$= A^{1000}(\alpha_1 x_1 + \cdots + \alpha_n x_n) / \lambda_1^{1000}$$

$$= (\alpha_1 \lambda_1^{1000} x_1 + \cdots + \alpha_n \lambda_n^{1000} x_n) / \lambda_1^{1000}$$

$$= \alpha_1 x_1 + \cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^{1000} x_n$$

In-class activity: Eigenvalues