$|\mu - \lambda_k| \leq \mathrm{cond}(X) \cdot \|E\|$

4.2 Properties and Transformations

What do the following transformations of the eigenvalue problem $A\mathbf{x} = \lambda \mathbf{x}$ do?

- **Shift.** $A \to A - \sigma I$ $\qquad (A - \sigma I)\vec{x} = (\lambda - \sigma)\vec{x}$
- **Inversion.** $A \to A^{-1}$ $\qquad A^{-1}x = \frac{1}{\lambda}x$
- **Power.** $A \to A^k$ $\qquad A^k x = \lambda^k x$
- **Polynomial** $A \to a\,A^2 + b\,A + c\,I$

$$(a\,A^2 + b\,A + c\,I)\mathbf{x} = (a\lambda^2 + b\lambda + c)\mathbf{x}$$

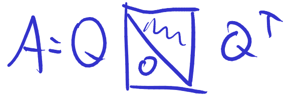
- **Similarity** $T^{-1}A\,T$ with T invertible

$B \coloneqq T^{-1}AT$

Schur form

- Show: Every matrix is orthonormally similar to an upper triangular matrix, i.e.

$$A = QUQ^T.$$



This is called the Schur form or Schur factorization.

Also, if we knew how to compute this, how would it help us find eigenvalues?

$A\vec{x} = \lambda\vec{x}$ $V = \mathrm{span}\{\vec{x}\}$

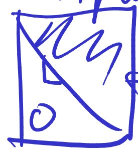
$A: V \to V$

$V^\perp \to V \oplus V^\perp$

$$A = \underbrace{\begin{pmatrix} | & & & \\ \vec{x} & \vec{x}_1 & \cdots & \vec{x}_{n-1} \\ | & & & \end{pmatrix}}_{Q_1} \begin{pmatrix} \lambda & \\ 0 & \\ 0 & X \\ 0 & \\ 0 & \end{pmatrix} Q_1^T$$

V $V^\perp$

if eigenvalue complex, but if we want real Schur form, 2x2 bump



4.3 Computing Eigenvalues

Power Iteration

- What are the eigenvalues of A^{1000}?

  Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Further assume $\|\boldsymbol{x}_i\| = 1$.

$$\vec{x} = \alpha_1 \vec{x}_1 + \cdots + \alpha_n \vec{x}_n$$

$$\frac{A^{1000}\vec{x}}{\lambda_1^{1000}} = \frac{\alpha_1 \cancel{\lambda_1^{1000}} \vec{x}_1}{\cancel{\lambda_1^{1000}}} + \alpha_2 \underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)}_{<1}^{1000} \vec{x}_2 + \cdots + \alpha_n \underbrace{\left(\frac{\lambda_n}{\lambda_1}\right)}_{<1}^{1000} \vec{x}_n$$

"Power iteration"

Problems:
- overflow (divide by $\|x\|$)
- $|\lambda_1| = |\lambda_2|$ → Problem
- $\alpha_1 = 0$ → $A^{1000}\vec{x}$ has no comp. in $\vec{x}_1$
  but: rounding error

Power Iteration: Issues?

- What could go wrong with Power Iteration?

What about Eigenvalues?

- Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

$\vec{x}$: approx eigenvector

$$\underbrace{\lambda \approx \frac{x^T A \vec{x}}{x^T \vec{x}}}$$

Rayleigh quotient

Convergence of Power Iteration

- What can you say about the convergence of the power method? Say $\boldsymbol{v}_1^{(k)}$ is the kth estimate of the eigenvector $\boldsymbol{x}_1$, and

$$e_k = \left\| \boldsymbol{x}_1 - \boldsymbol{v}_1^{(k)} \right\|.$$

$\tilde{x}_0$

$x_1 := A x_0$

$x_2 := A x_1 = A^2 x_0$

$x_{k+1} = A x_k$

$$e_{k+1} \approx \frac{|\lambda_2|}{|\lambda_1|} \cdot e_k$$

← linear convergence

too slow

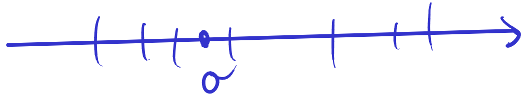
With shift: $$e_{k+1} \approx \frac{|\lambda_2 - \sigma|}{|\lambda_1 - \sigma|} \cdot e_k$$

With shift and inversion $$e_{k+1} \approx \frac{|\lambda_1 - \sigma|}{|\lambda_2 - \sigma|} \cdot e_k$$

Rayleigh Quotient Iteration

- Describe inverse iteration.

$$x_{k+1} := (A-\sigma)^{-1} x_k$$



"new σ"

amplifies eigenvalue that's closest to sigma

- ~~Describe Rayleigh Quotient Iteration~~.

$$e_{k+1} \approx \left| \frac{\lambda_{\text{closest}} - \sigma}{\lambda_{\text{second-closest}} - \sigma} \right| e_k$$

Pick $\sigma = \dfrac{x^T A x}{x^T x}$, then this Rayleigh quotient iteration.

Demo: Power Iteration and its Variants

Computing Multiple Eigenvalues

- All Power Iteration Methods compute one eigenvalue at a time.
  What if I want *all* eigenvalues?