## QR Iteration/QR Algorithm

Orthogonal iteration:

$X_0 = A$

$Q_k R_k = X_k$

$X_{k+1} = A\, Q_k$

QR iteration:

$\bar{X}_0 = A$

$\bar{Q}_k \bar{R}_k = \bar{X}_k$

$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k$

n times

Tracing through reveals:

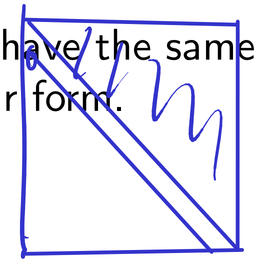
- $\hat{X}_k = \bar{X}_{k+1}$
- $Q_0 = \bar{Q}_0$
  $Q_1 = \bar{Q}_0 \bar{Q}_1$
  $Q_k = \bar{Q}_0 \bar{Q}_1 \cdots \bar{Q}_k$

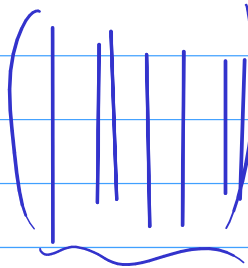
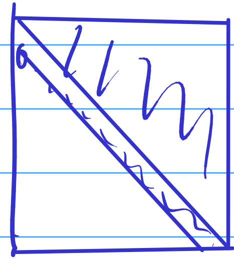
- conv. slowly
- expensive

Orthogonal iteration showed: $\hat{X}_k = \bar{X}_{k+1}$ converge. Also:

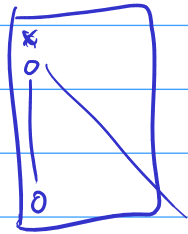
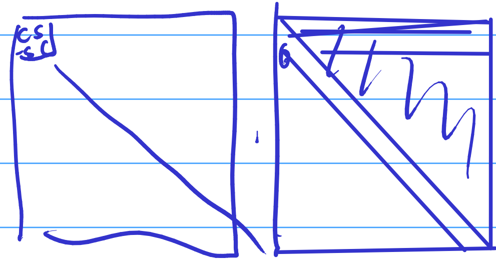
$$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k = \bar{Q}_k^T \bar{X}_k \bar{Q}_k,$$

so the $\bar{X}_k$ are all similar $\rightarrow$ all have the same eigenvalues.
$\rightarrow$ QR iteration produces Schur form.



 $= QR$ 

$A$

$I - 2\frac{vv^T}{v^Tv}$ $\quad H_n \cdots H_1 A =$  

1 Householder: $O(n^2)$ — 1 Givens rot $O(n)$

n Householders $O(n^3)$ — n Givens rot $O(n^2)$

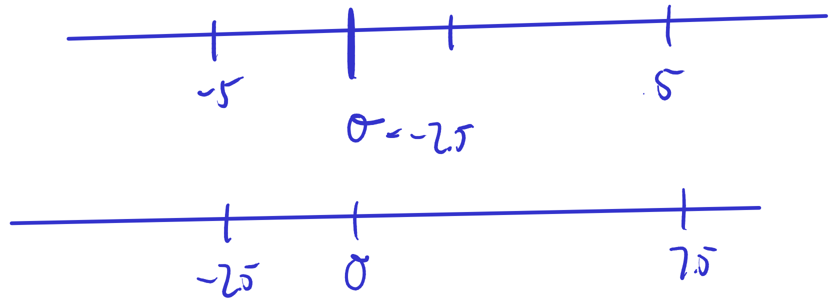
## QR Iteration: Incorporating a Shift

- How can we accelerate convergence of QR iteration using shifts?

$$\bar{X}_0 = A$$

$$\bar{Q}_k \bar{R}_k = \bar{X}_k - \sigma_k I_n$$

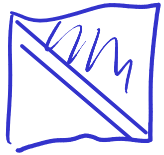
$$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k + \sigma_k I$$



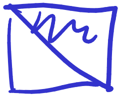
## QR Iteration: Computational Expense

- A full QR factorization at each iteration costs $O(n^3)$–can we make that cheaper?

Upper Hessenberg form

$A = Q$  $Q^T$

By getting A into upper H. form before we start QR iteration, we can get the cost of 1 step of QR it. to $O(n^2)$.

$A = Q$  $Q^T$

**Demo:** Householder Similarity Transforms

## 4.4 Krylov Space Methods

QR it.

span $[A^\ell y_1, A^\ell y_2, \dots, A^\ell y_n]$

Krylov space

$$[\underbrace{x}_{x_0}, \underbrace{Ax}_{x_1}, \underbrace{A^2x}_{x_2}, A^3x, \dots, A^{n-1}x]$$

$$K_k = \begin{pmatrix} | & & | \\ x_0 & \cdots & x_{k-1} \\ | & & | \end{pmatrix} \quad (n \times k) \qquad (1 \le k \le n)$$

$$AK_k = \begin{pmatrix} | & | & & | & | \\ x_1 & x_2 & \cdots & x_{k-1} & x_k \\ | & | & & | & | \end{pmatrix} = K_k \underbrace{\begin{pmatrix} 0 & 0 & & 0 & | \\ 1 & 0 & & & K_n^{-1}x_k \\ 0 & 1 & \ddots & & | \\ \vdots & 0 & & 1 & | \\ & \vdots & & & \end{pmatrix}}_{C_n}$$

$$A K_n = K_n C_n$$

$$K_n^{-1} A K_n = C_n$$

## Krylov space methods: Intro

- What subspaces can we use to look for eigenvectors?

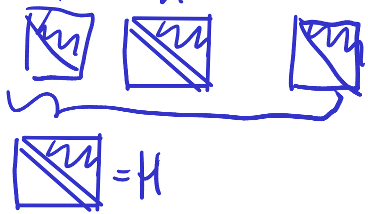
## Conditioning in Krylov Space Methods/Arnoldi Iteration

- What is a problem with Krylov space methods? How can we fix it?

$$Q_n R_n = K_n \quad \Rightarrow \quad Q_n = K_n R_n^{-1}$$

Then:

$$Q_n^T A Q_n = R_n \underbrace{K_n^{-1} A K_n}_{C_n} R_n^{-1}$$



$= H$

$$Q_n^T A Q_n = H$$

$$Q_n = \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix}$$

$$A Q_n = Q_n H$$

$$A q_k = h_{1k} \vec{q}_1 + \cdots + h_{k+1\,k} \vec{q}_{k+1}$$

$$H = (h_{ij})$$

$$q_j^T A q_k = h_{jk}$$

$$q_j^T q_\ell = \begin{cases} 1 & j = \ell \\ 0 & j \neq \ell \end{cases}$$

**Demo:** Arnoldi Iteration (Part I)

## Krylov: What about eigenvalues?

- How can we use Arnoldi/Lancos to compute eigenvalues?