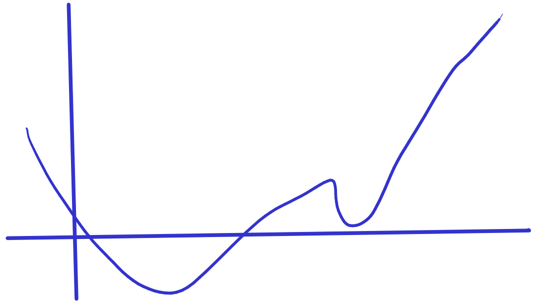
Showing Existence

- How can we show existence of a root?

- IVT
- Inv FT
- CMT

$$\|g(x)-g(y)\| \le \gamma \|x-y\|$$

$0<\gamma<1$



$\rightarrow f(x)=0$

$g(x)=x$

$g(x)=f(x)+x$

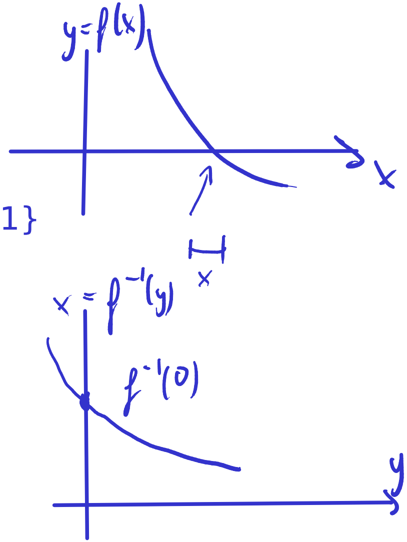
Uniqueness? (um, no.)

Sensitivity and Multiplicity

- What is the sensitivity/conditioning of root finding?

$f^{-1}(0) \to$

conditioning of root finding
is the same as conditioning
of function evaluation of f^{-1}
at 0.



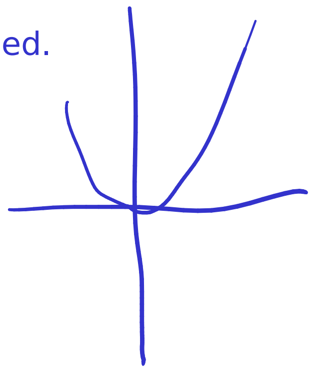
- What are multiple roots?

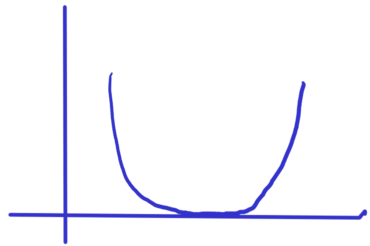
A root of multiplicity m $\qquad (x-\lambda)^m$

$f(x)=0 \quad f'(x)=0 \quad \dots \quad f^{(m-1)}(x)=0$

- How do multiple roots interact with conditioning?

This implies that root finding
(for that root) is poorly conditioned.





5.1 Iterative Procedures

- What is linear convergence? quadratic convergence?

$$e_k = x_k - x^*$$

An iterative method converges with rate r if

$$\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C \begin{cases} >0 \\ <\infty \end{cases}$$

$r>1$ superlinear

$r=1$: linear conv. → eg. power iteration

"gains a constant number of digits every iteration"

$r=2$: quadratic conv. -> e.g. Rayleigh quotient it.

"doubles the number of digits every iteration"

About Convergence Rates

Demo: Rates of Convergence

- Characterize linear, quadratice convergence in terms of the 'number of accurate digits'.

Convergence Rates: Understanding the Definition

- Contrast the following 'alternate' definitions of convergence rate:

$$(1) \quad \frac{\|\boldsymbol{e}_{k+1}\|}{\|\boldsymbol{e}_k\|^r} \leqslant C \begin{cases} >0, \\ <\infty. \end{cases}$$

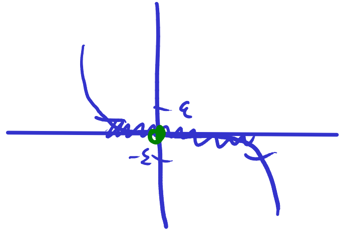
$$(2) \quad 0 < C_{\text{low}} \leqslant \frac{\|\boldsymbol{e}_{k+1}\|}{\|\boldsymbol{e}_k\|^r} \leqslant C_{\text{high}}$$

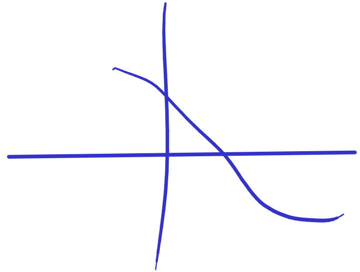
$$(3) \quad \lim_{k\to\infty} \frac{\|\boldsymbol{e}_{k+1}\|}{\|\boldsymbol{e}_k\|^r} = C \begin{cases} >0, \\ <\infty. \end{cases}$$

Stopping Criteria

- Here are some ideas for stopping criteria:
  1. $|f(x)| < \varepsilon$ ('residual is small')
  2. $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| < \varepsilon$
  3. $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| / \|\boldsymbol{x}_k\| < \varepsilon$

  Comment on them. Are any of them 'foolproof'?

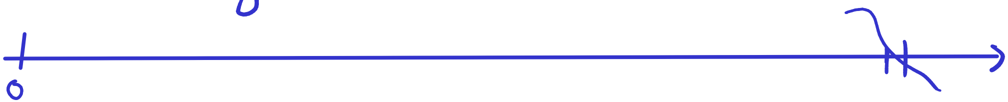




1. might be off if f is shallow near the root.

2.

3. near 0: also bad
has nothing to do with being close to $f(x)=0$.



5.2 Methods in One Dimension

Bisection Method

Demo: Bisection Method

- What's the rate of convergence? What's the constant?

1

1/2

Fixed Point Iteration

$$x_0 = \langle \text{starting guess} \rangle$$
$$x_{k+1} = g(x_k)$$

Demo: Fixed point iteration

- When does fixed point iteration converge? Assume g is smooth.

↑ if g is contractive

$g(x^*) = x^*$

$$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$$
$$\approx g'(x^*) \underbrace{(x_k - x^*)}_{e_k}$$

$$\frac{|e_{k+1}|}{|e_k|} \approx |g'(x^*)|$$

Criterion: If $|g'(x^*)| < 1$ then FPI converges (locally).

In-class activity: Nonlinear equations in 1D

Newton's Method

- Derive Newton's method.