Fixed Point Iteration

$$f(x^*) = 0 \rightarrow g(x) = f(x) + x$$

$$g(x^*) = x^*$$

$$\begin{aligned} x_0 &= \langle \text{starting guess} \rangle \\ x_{k+1} &= g(x_k) \end{aligned}$$

Demo: Fixed point iteration

- When does fixed point iteration converge? Assume g is smooth.

$$x^* = g(x^*)$$

$$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$$

$$\approx g'(x^*)(x_k - x^*) = \underbrace{g'(x^*)}_{|\cdot|<1} e_k$$

If $g'(x^*) = 0$, Taylor gives

$$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*) \approx g''(x^*)(x_k - x^*)^2/2$$

Newton's Method $\leftarrow$

- Derive Newton's method. $f(x) \overset{!}{=} 0$ Can eval f, f'

$$f(x_k + h) \approx f(x_k) + h f'(x_k) = \hat{f}(h)$$

$$0 = \hat{f}(h) = f(x_k) + h f'(x_k)$$

$$-f(x_k) = h f'(x_k) \quad \rightsquigarrow \quad h = -\frac{f(x_k)}{f'(x_k)}$$

$$x_{k+1} = x_k + h = \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{g(x_k)}$$

$$\underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{g(x_k)}$$

$$g'(x) = \frac{f(x) f''(x)}{f'(x)^2}$$

Suppose x^* is $f(x^*) = 0$.

$$g(x^*) = x^*$$

Disclaimer: Assume $f'(x_k) \neq 0$.

Quadratic conv. if $g'(x^*) = 0$.

$$g'(x^*) = \frac{\overset{0}{\cancel{f(x^*)}} f''(x^*)}{f'(x^*)}$$

For single roots: quadratic convergence (at least locally)

For multiple roots: linear conv.

Demo: Newton's method
Demo: Convergence of Newton's Method

Secant Method

- What would Newton without the use of the derivative look like?

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$x_{k+1} = x_k + h = x_k - \frac{f(x_k)}{\cancel{f'(x_k)} \quad \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}}$$

$$\frac{e_2}{e_1^r} \approx C$$

Demo: Secant Method
Demo: Convergence of the Secant Method
In-class activity: Nonlinear equations in 1D

‘Trusting’ Newton and Secant

- The linear approximations in Newton and Secant are only good locally. How could we use that?

5.3 Methods in n Dimensions ("Systems of Equations")

Fixed Point Iteration

$$\begin{aligned} \boldsymbol{x}_0 &= \langle \text{starting guess} \rangle \\ \boldsymbol{x}_{k+1} &= \boldsymbol{g}(\boldsymbol{x}_k) \end{aligned}$$

- When does this converge?