Lagrange Multipliers

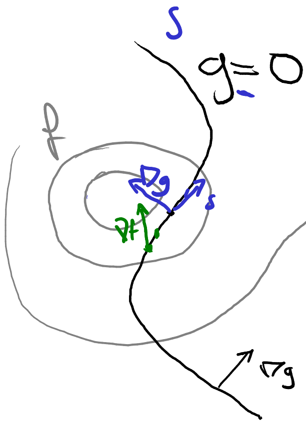


Figure out necessary cond.

- $g(\vec{x})=0$
- $\nabla g \cdot \vec{s} = 0$
- At the min

$$\nabla f = \lambda \cdot \nabla g$$

$$\mathcal{L}(x, \lambda) = f(x) + \lambda \cdot g(x)$$

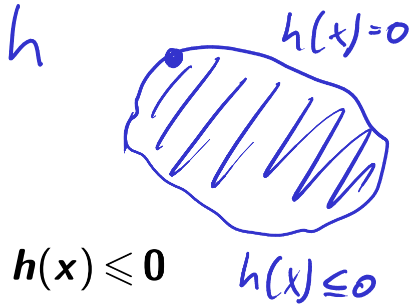
○ Seen: Need

$$-\nabla f(\boldsymbol{x}) = J_{\boldsymbol{g}}^{T} \boldsymbol{\lambda}$$

at the (constrained) optimum.

Idea: Turn constrained optimization problem for $\boldsymbol{x}$ into an *unconstrained* optimization problem for $(\boldsymbol{x}, \boldsymbol{\lambda})$. How?

Demo: Sequential Quadratic Programming



Inequality-Constrained Optimization

Want $\boldsymbol{x}^*$ so that

$$f(\boldsymbol{x}^*) = \min_{\boldsymbol{x}} f(\boldsymbol{x}) \quad \text{subject to} \quad \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{h}(\boldsymbol{x}) \leqslant \boldsymbol{0}$$

This is inequality-constrained optimization. Develop a necessary condition for a minimum.

Define Lagrangian:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) := f(\boldsymbol{x}) + \boldsymbol{\lambda}_1^T \boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{\lambda}_2^T \boldsymbol{h}(\boldsymbol{x})$$

- Some inequality constrains may not be "active"
  (active $\Leftrightarrow h_i(\boldsymbol{x}^*) = 0 \Leftrightarrow$ at 'boundary' of ineq. constraint)
  (Equality constrains are always 'active')
- If h_i inactive ($h_i(\boldsymbol{x}^*) < 0$), must force $\lambda_{2,i} = 0$.

Otherwise: Behavior of h could change location of minimum of $\mathcal{L}$.
Use complementarity condition

$$h_i(\boldsymbol{x}^*)\lambda_{2,i} = 0.$$

Assuming $J_{\boldsymbol{g}}$ and $J_{\boldsymbol{h},\text{active}}$ have full rank, this set of conditions is *necessary*:

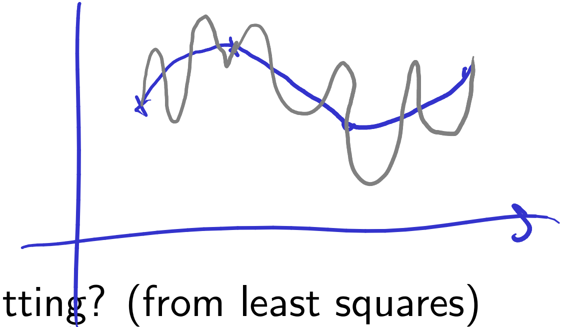
$$\left.\begin{array}{rrcl}
(*) & \nabla_{\boldsymbol{x}}\mathcal{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}_1^*, \boldsymbol{\lambda}_2^*) & = & \mathbf{0} \\
(*) & \boldsymbol{g}(\boldsymbol{x}^*) & = & \mathbf{0} \\
 & \boldsymbol{h}(\boldsymbol{x}^*) & \leqslant & \mathbf{0} \\
 & \boldsymbol{\lambda}_2 & \geqslant & \mathbf{0} \\
(*) & \boldsymbol{h}(\boldsymbol{x}^*) \cdot \boldsymbol{\lambda}_2 & = & 0
\end{array}\right\}$$

These are called the Karush-Kuhn-Tucker ('KKT') conditions.

Computational approach: Solve $(*)$ equations by Newton.

7 Interpolation

Interpolation: Setup



Given: $(x_i)_{i=1}^N$, $(y_i)_{i=1}^N$

Wanted: Function f so that $f(x_i) = y_i$

- How is this not the same as function fitting? (from least squares)

  because: we exactly hit the values

- Does this problem have a unique answer?

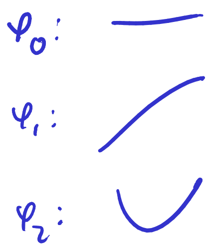
- Why is this important?

  because: calculus.

Making the Interpolation Problem Unique

- How can we cut down the set of possible answers to *exactly one*?

$$f(x) = \sum_{j=1}^{N_{func}} \alpha_j \varphi_j(x)$$



$$y_i = f(x_i) = \sum_{j=1}^{N_{func}} \alpha_j \underbrace{\varphi_j(x_i)}_{V_{ij}} \Leftrightarrow V\vec{\alpha} = \vec{y}$$

↳ generalized Vandermonde matrix

V(coefficient vec.) = (values $\vec{y}$ at interp. nodes (x_i))

conditioning of interpolation: same as solving $V\vec{\alpha} = \vec{y}$

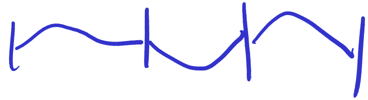
Modes and Nodes (aka Functions and Points)

Both function basis and point set are under our control. What do we pick?

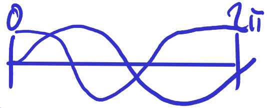
Ideas for functions:

- Monomials $1, x, x^2, x^3, x^4, \ldots$

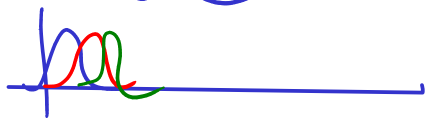
- Functions that make $\boldsymbol{V} = \boldsymbol{I} \to$ 'Lagrange basis'
- Functions that make $\boldsymbol{V}$ triangular $\to$ 'Newton basis'
- Splines (piecewise polynomials)



- Orthogonal polynomials
- Sines and cosines



- 'Bumps' ('Radial Basis Functions')



Ideas for points:

- Equispaced

- ‘Edge-Clustered’ (so-called Chebyshev/Gauss/... nodes)

◦ But first: Why *not* monomials on equispaced points?

◦ Why not equispaced?

$$\sum_j \alpha_j \underbrace{x_i^j}_{V_{ij}} = y_i$$

$$V = \begin{pmatrix} x_0^0 & \cdots & x_0^n \\ \vdots & & \\ x_n^0 & \cdots & x_n^n \end{pmatrix}$$

$$x_i = \begin{pmatrix} \vdots \\ \hline n \end{pmatrix}$$

Lagrange Interpolation

- Find a basis so that $V = I$, i.e.

$$\varphi_j(x_i) = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$