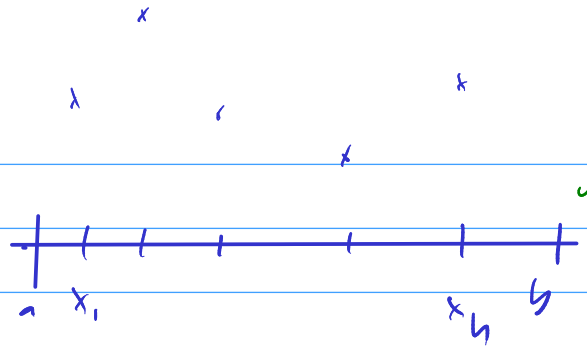

$f(x) = \alpha_1 \varphi_1(x) + \cdots + \alpha_n \varphi_n(x)$

$f'(x) = \alpha_1 \varphi_1'(x) + \cdots + \alpha_n \varphi_n'(x)$

$$\left.\begin{aligned} \alpha_1 \varphi_1(x_1) + \cdots + \alpha_n \varphi_n(x_1) &= y_1 \\ &\ddots \\ \alpha_1 \varphi_1(x_n) + \cdots + \alpha_n \varphi_n(x_n) &= y_n \end{aligned}\right\} \quad V\vec{\alpha} = \vec{y}$$

$$\begin{pmatrix} \varphi_1(x_1) & \varphi_2(x_1) \cdots \\ \varphi_1(x_2) & \\ \vdots & \end{pmatrix}$$

Modes and Nodes (aka Functions and Points)

Both function basis and point set are under our control. What do we pick?

Ideas for functions:

- Monomials $1, x, x^2, x^3, x^4, \ldots$ ✓
- Functions that make $V = I \rightarrow$ 'Lagrange basis'
- Functions that make V triangular $\rightarrow$ 'Newton basis'
- Splines (piecewise polynomials)
- Orthogonal polynomials
- Sines and cosines
- 'Bumps' ('Radial Basis Functions')

Ideas for points:

- Equispaced ✓

- 'Edge-Clustered' (so-called Chebyshev/Gauss/... nodes)

- But first: Why *not* monomials on equispaced points?
- Why not equispaced?

Lagrange Interpolation

- Find a basis so that $V = I$, i.e.

$$\varphi_j(x_i) = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

$x_1 \quad x_2 \quad x_3$

$$\varphi_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$\varphi_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$\varphi_3(x) =$$

Lagrange Polynomials: General Form

$$\varphi_j(x) = \frac{\prod_{k=1, k\neq j}^{m} (x - x_k)}{\prod_{k=1, k\neq j}^{m} (x_j - x_k)}$$

Newton Interpolation

- Find a basis so that V is triangular.

$$\varphi_1(x) = 1$$
$$\varphi_2(x) = (x - x_1)$$
$$\varphi_3(x) = (x - x_1)(x - x_2)$$

- Why not Lagrange/Newton?

↳ solve with BU / FW substitution

$V \triangleleft$

↳ Another process:

"Divided Differences"

Better conditioning: Orthogonal polynomials

- What caused monomials to have a terribly conditioned Vandermonde?
- What's a way to make sure two vectors are *not* like that?
- But polynomials are functions!
- But how can I practically compute the Legendre polynomials?

$f(x) \perp g(x)$?

$\int_a^b f(x)\, g(x)\, dx = 0$

Inner product (f, g)

$\vec{f} \perp \vec{g}$

$\sum f_i \cdot g_i = 0$

$$(f, g) = \int f(x)\, g(x)\, w(x)\, dx$$

Another family of orthogonal polynomials: Chebyshev

Three equivalent definitions:

- Result of Gram-Schmidt with weight $1/(1-x^2)$
  - What is that weight?

  (Again, you won't exactly get the standard normalization if you do this.)
- $T_k(x) = \cos(k \cos^{-1}(x))$
- $T_k(x) = 2x\,T_k(x) - T_{k-1}(x)$

Demo: Chebyshev interpolation part I

- What is the Vandermonde matrix for Chebyshev polynomials?

Chebyshev nodes

Might also consider zeros (instead of roots) of T_k:

$$x_i = \cos\left(\frac{2i+1}{2k}\pi\right) \quad (i = 1\ldots, k).$$

The Vandermonde for these (with T_k) can be applied in $O(N \log N)$ time, too.

It turns out that we were still looking for a good set of interpolation nodes.

- We came up with the criterion that the nodes should bunch towards the ends. Do these do that?

Demo: Chebyshev interpolation part II

- Summary?

Error Result

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!}(x - x_1)(x - x_2) \cdots (x - x_n)$$

- Why does Chebyshev-like 'bunching' work?