8 Numerical Integration and Differentiation

8.1 Numerical Integration

Numerical Integration: About the Problem

- What is numerical integration? (Or 'quadrature'?)

  $$\int_a^b f(x)\,dx$$

  a, b, f

- What about existence and uniqueness?

  - Existence: f is $\underbrace{\text{integrable}}_{\substack{\text{Riemann} \\ \text{Lebesgue}}}$

  - Piecewise continuous

Conditioning

- Derive the (absolute) condition number for numerical integration.

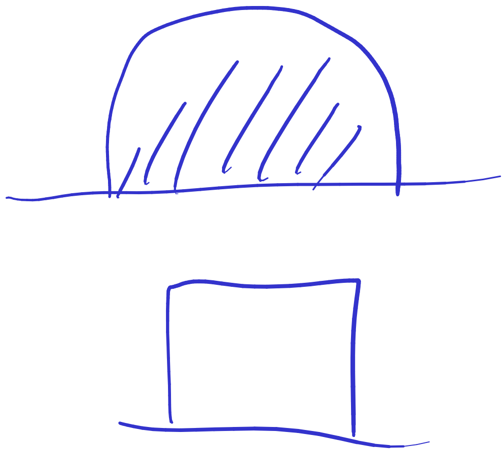
$$\hat{f}(x) = f(x) + e(x)$$

$$\left| \int f(x)\,dx - \int \hat{f}(x)\,dx \right|$$

$$= \left| \int e(x)\,dx \right| \leq \int |e(x)|\,dx$$

$$\leq (b-a) \max_{x \in [a,b]} |e(x)| \quad \leftarrow \text{absolute cond. number}$$

8.1.1 Quadrature Methods



Interpolatory Quadrature

- Design a quadrature method based on interpolation.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \omega_i f(x_i)$$

Interpolate at (x_i):

$$f(x) \approx \sum_{i=1}^{n} f(x_i)\,\ell_i(x)$$

Lagrange Polynomial for x_i

Integrate:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i) \underbrace{\int_a^b \ell_i(x)\,dx}_{\omega_i}$$

- With monomials and equispaced pts. this is called Newton-Cotes quadrature
- With Chebyshev nodes and Chebyshev polynomials, this is called Clenshaw-Curtis quadrature.

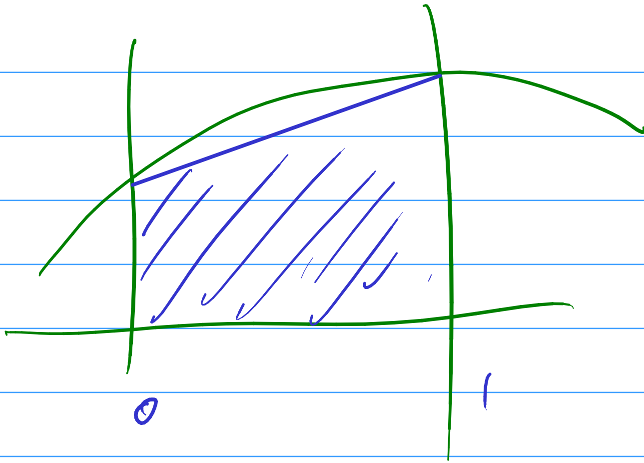
Method of undet. coefficients

$$b-a = \int_a^b 1\,dx = w_1 \cdot 1 + \dots + w_n \cdot 1$$

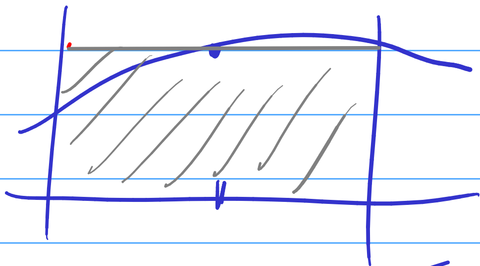
$$\frac{1}{2}(b^2-a^2) = \int_a^b x\,dx = w_1 \cdot x_1 + \dots + w_n \cdot x_n$$

$$\vdots$$

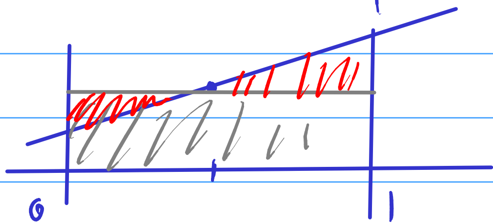
$$\frac{1}{k+1}(b^{k+1}-a^{k+1}) = \int_a^b x^k\,dx = w_1 \cdot x_1^k + \dots + w_n \cdot x_n^k$$



$\frac{1}{2}(f(0)+f(1))$



$1 \cdot f(0.5)$



$$\int_0^2 f(x)\,dx = \int_0^1 f(2\tau)\underbrace{\frac{dx}{d\tau}}_{2}d\tau$$

$$\tau = x/2$$

$$dx = 2d\tau$$

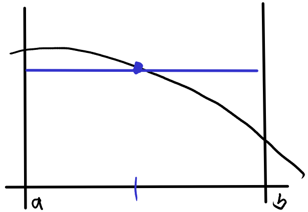
Demo: Newton-Cotes weight finder

Examples and Exactness

- To what polynomial degree are the following rules exact?

Midpoint rule

$$(b-a)f\left(\frac{a+b}{2}\right)$$



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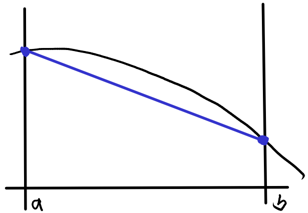
Trapezoidal rule

$$\frac{b-a}{2}(f(a)+f(b))$$



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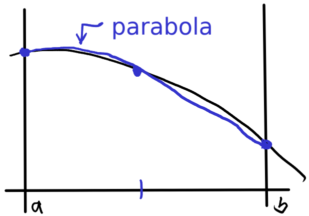
Simpson's rule

$$\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$



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8.1.2 Accuracy and Stability

Interpolatory Quadrature: Accuracy

- Let p_{n-1} be an interpolant of f at nodes $x_1, \ldots, x_n$ (of degree $n-1$)
  Recall

$$\sum_i \omega_i f(x_i) = \int_a^b p_{n-1}(x)\mathrm{d}x.$$

  What can you say about the accuracy of the method?

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

$$\left|\int f(x)dx - \sum_{i=1}^{n} w_i f(x_i)\right| = \left|\int f(x)dx - \int p_{n-1}(x)dx\right|$$

$$\le \int_a^b |f(x)dx - \int p_{n-1}(x)|dx$$

$$\le (b-a)\|f-p_{n-1}\|_\infty$$

$$\le \underbrace{(b-a)}_{h} C\cdot\|f^{(n)}\|_\infty h^n = C\cdot\|f^{(n)}\|_\infty \cdot h^{n+1}$$

From interpolation:

$$\|f-p_{n-1}\|_\infty \le C\cdot\|f^{(n)}\|_\infty h^n$$

Demo: Accuracy of Simpson's rule

Interpolatory Quadrature: Stability

- Let p_n be an interpolant of f at nodes $x_1, \dots, x_n$ (of degree $n-1$)
  Recall
  $$\sum_i \omega_i f(x_i) = \int_a^b p_n(x)\mathrm{d}x$$
  What can you say about the stability of this method?