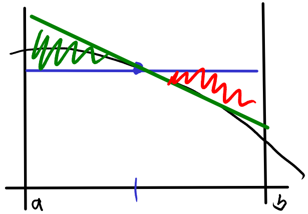
Examples and Exactness

- To what polynomial degree are the following rules exact?

exact to: degree   o.o.a.

Midpoint rule

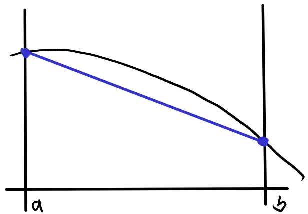
$$(b-a)f\left(\frac{a+b}{2}\right)$$



~~0~~ 1   3

Trapezoidal rule

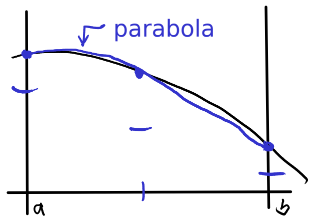
$$\frac{b-a}{2}(f(a)+f(b))$$



1   3

Simpson's rule

$$\frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$



~~2~~ 3   5

n # pts   Interpolation error est: h^n

Quadrature error est: h^{n+1}

8.1.2 Accuracy and Stability

Interpolatory Quadrature: Accuracy

- Let p_{n-1} be an interpolant of f at nodes $x_1, \ldots, x_n$ (of degree $n-1$)
  Recall

$$\sum_i \omega_i f(x_i) = \int_a^b p_{n-1}(x)\mathrm{d}x.$$

  What can you say about the accuracy of the method?

Demo: Accuracy of Newton-Cotes

Interpolatory Quadrature: Stability

- Let p_n be an interpolant of f at nodes $x_1, \ldots, x_n$ (of degree $n-1$)
  Recall

$$\sum_i \omega_i f(x_i) = \int_a^b p_n(x) \mathrm{d}x$$

  What can you say about the stability of this method?

Consider $\hat{f}(x) = f(x) + e(x)$

$$\left| \sum_i \omega_i f(x_i) - \sum_i \omega_i \hat{f}(x_i) \right| = \left| \sum_i \omega_i e(x_i) \right|$$

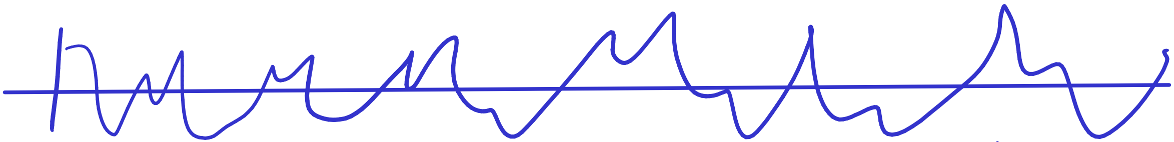
$$\leq \underbrace{\|e\|_\infty}_{} \sum_i |\omega_i|$$

$\sum \omega_i = b-a$

About Newton-Cotes

- What's not to like about Newton-Cotes quadrature?

  - stability for high point counts: ✗
  - Vdm is ill-conditioned: ✗
  - inherits all issues from high-order poly interp.
  - hard to extend to high # points

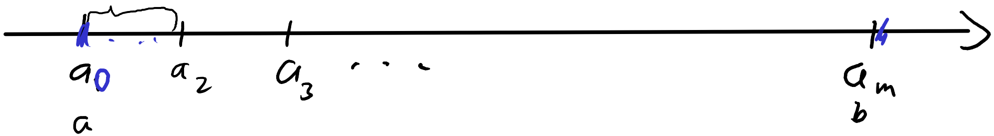


Ideas: Lots of tiny sub-quadratures "Composite quad."

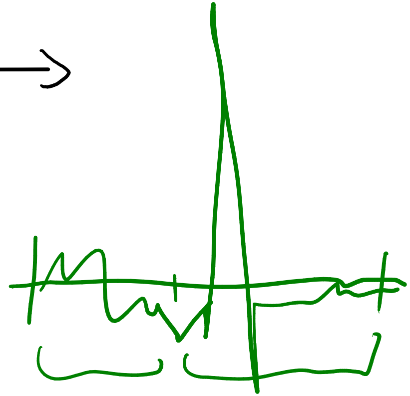
Make high-order work — Gaussian quadrature

8.1.3 Composite Quadrature

- High-order polynomial interpolation requires a high degree of smoothness of the function.
  Idea: Stitch together multiple lower-order quadrature rules to alleviate smoothness requirement.
  e.g. trapezoidal



What can we say about the error in this case?



Single interv: $\left|\int_0^h (f - p_{n-1})\right| \le \underline{C\cdot h^{n+1}\|f^{(n)}\|_\infty}$

Multi-interv: $\left|\int_a^b f(x)\,dx - \sum_{j=1}^{m}\sum_{i=1}^{n} \omega_{j,i}\, f(x_{i,j})\right|$

$$\le C\cdot\|f^{(n)}\|_\infty \sum_{j=1}^{m} (\underbrace{a_j - a_{j-1}}_{\le h})^{n+1}$$

$$\le C\cdot\|f^{(n)}\|_\infty \sum_j h^n (a_j - a_{j-1}) = \underline{C\|f^{(n)}\|_\infty h^n}\cdot\overbrace{\sum_j (a_j - a_{j-1})}^{b-a}$$

8.1.4 Gaussian Quadrature

- So far: nodes chosen from outside.
  Can we gain something if we let the quadrature rule choose the weights, too? **Hope:** More design freedom $\rightarrow$ Exact to higher degree.

Demo: Gaussian quadrature weight finder

$$\int_a^b x^k \, dx = \alpha_1 \cdot x^k + \cdots + \alpha_n x^k$$

8.2 Numerical Differentiation